CSCI 6220/4030: Homework 3

Assigned Thursday October 12 2017. Due at beginning of class Thursday Oct 26 2017.

Remember to typeset your submission, and label it with your name. Please start early so you have ample time to see me during office hours. Provide mathematically convincing arguments for the following problems. Ask me if you are unclear whether your arguments are acceptable.

1. Recall that the XOR of a collection of Boolean literals is 1 if an odd number of the literals are true, and 0 otherwise:

$$\text{XOR}(\{Y_i\}_{i \in S}) = \begin{cases} 1 & \text{an odd number of the } Y_i \text{ are true} \\ 0 & \text{otherwise.} \end{cases}$$

   Let $Y_1, \ldots, Y_m$ be independent Boolean r.v.s that are equally likely to be true or false. Given S, a non-empty subset of $\{1, \ldots, m\}$, define the random variable $X_S = \text{XOR}(\{Y_i\}_{i \in S})$. Show that the $2^m - 1$ random variables constructed in this manner are pairwise independent $\text{Bern}(\frac{1}{2})$ r.v.s, but are not mutually independent.

2. Recall that K_n is the complete graph on n vertices. Use the Lovász Local Lemma to show that, if

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1,$$

   then it is possible to color the edges of K_n with two colors so that it has no monochromatic K_k subgraph.

3. We have shown using the probabilistic method that, if a graph G has n vertices and m edges, then there exists a cut with cost at least $m/2$. Improve this result by showing that there exists a cut with cost at least $mn/(2n-1)$.

4. An undirected graph on n vertices is disconnected if there exists a set of $k < n$ vertices such that there is no edge between this set and the rest of the graph. Otherwise, the graph is said to be connected. Show that there exists a constant c such that if $N \geq cn \log n$, then with probability $1 - o(1)$, a graph randomly chosen from $G_{n,N}$ is connected.

   Here, $G_{n,N}$ is the collection of graphs with n vertices that have exactly N edges selected randomly (without replacement) from the set of $\binom{n}{2}$ possible edges.