1. Two rooted trees $T_1$ and $T_2$ are said to be isomorphic if there exists a one-to-one mapping $f$ from the vertices of $T_1$ to those of $T_2$ satisfying the following condition: for each internal vertex $v$ of $T_1$ with the children $v_1, \ldots, v_k$, the vertex $f(v)$ has as children exactly the vertices $f(v_1), \ldots, f(v_k)$. Observe that no ordering is assumed on the children of any internal vertex. Devise and efficient randomized algorithm for testing the isomorphism of rooted trees using the Schwarz-Zippel theorem, and analyze its cost and probability of success. Hint: associate a polynomial $P_v$ with each vertex $v$ in a tree $T$. The polynomials are defined recursively, with the base case being that the leaf vertices all have $P = x_0$. An internal vertex $v$ of height $h$ with the children $v_1, \ldots, v_k$ has its polynomial defined to be $(x^h - P_{v_1}) \cdots (x^h - P_{v_k})$. Note that there is one indeterminate $x^h$ associated with each level $h$ in the tree.

2. Consider two computers, each containing $n$ bit strings of length $n$. It can be shown that any deterministic algorithm for determining whether these sets have a non-empty intersection requires $O(n^2)$ bits to be communicated between the computers. Design a Las Vegas algorithm for answering this problem that communicates $O(n \log n)$ bits in expectation.

3. Gas molecules move about randomly in a box that is divided into two halves symmetrically by a partition; there is a hole in the partition. Suppose there are $n$ molecules in the box. Molecular motion can be modeled by choosing a number between 1 and $n$ at random and moving the corresponding molecule from one side of the partition to the other.

   (a) Show that the number of molecules on one side of the partition evolves as a Markov Chain. What are the states, and the transition probabilities?

   (b) Argue that the chain is ergodic, and find its stationary distribution

   (c) Find the lowest upper-bound on $t_{\text{mix}}(\varepsilon)$ (starting from an arbitrary distribution) that you can.

4. Given a deck of $n$ cards in an arbitrary starting order, consider the following shuffling algorithm: at each step choose one of the cards uniformly at random (including the top card) and move it to the top of the deck. Identify the cards with the numbers $1, \ldots, n$ and argue that the invariant distribution of this Markov Chain is the uniform distribution over $S_n$, and that $t_{\text{mix}}(\varepsilon) \leq n \ln n + n \ln(1/\varepsilon)$. Hint: to construct the coupling, when you pick a card to move to the top of the X process, pick the same card to move to the top of the Y process.