CSCI 6220/4030: Homework 1

Assigned Monday September 10 2018. Due at beginning of class Thursday September 20 2017.

Provide well-written, clear, and convincing arguments for the following problems. Remember to typeset your submission, and label it with your name. Please start early so you have ample time to see me during office hours.

1. Establish that
$$\mathbb{P}[\cup_{i=1}^n \mathcal{E}_i] \leq \sum_{i=1}^n \mathbb{P}[\mathcal{E}_i]$$
and give a simple argument that this implies
$$\mathbb{P}[\cap_{i=1}^n \mathcal{E}_i] \geq 1 - \sum_{i=1}^n \mathbb{P}[\mathcal{E}_i^c].$$

2. Show that
$$\mathbb{P}[\cap_{i=1}^n \mathcal{E}_i] = \mathbb{P}[\mathcal{E}_n | \cap_{i=1}^{n-1} \mathcal{E}_i]\, \mathbb{P}[\mathcal{E}_{n-1} | \cap_{i=1}^{n-2} \mathcal{E}_i] \cdots \mathbb{P}[\mathcal{E}_2 | \mathcal{E}_1]\, \mathbb{P}[\mathcal{E}_1].$$

3. The probability of any of several events happening can be determined by using the principle of inclusion and exclusion: sum the probability of the events, then subtract the probability of the intersections of any pairs of events to avoid overcounting, then add the probability of the intersections of triplets of the events to avoid undercounting, and continue in this manner on up to the probability of the intersection of all of the events. This gives the Principle of Inclusion and Exclusion formula
$$\begin{aligned}\mathbb{P}[\cup_{i=1}^n \mathcal{E}_i] &= \sum_{i=1}^n \mathbb{P}[\mathcal{E}_i] - \sum_{1 \leq i < j \leq n} \mathbb{P}[\mathcal{E}_i \cap \mathcal{E}_j] \\ &+ \sum_{1 \leq i < j < k \leq n} \mathbb{P}[\mathcal{E}_i \cap \mathcal{E}_j \cap \mathcal{E}_k] - \cdots + (-1)^{n+1}\, \mathbb{P}[\cap_{i=1}^n \mathcal{E}_i] \\ &= \sum_{k=1}^n (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbb{P}[\mathcal{E}_{i_1} \cap \cdots \cap \mathcal{E}_{i_k}].\end{aligned}$$
Suppose that we want to know the probability that exactly k of the n events occur. Use the PIE formula to show that if N_k is the event that exactly k of the events occur, then
$$\mathbb{P}[N_k] = \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{k} S_{k+i}, \quad \text{where } S_j = \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq n} \mathbb{P}[\mathcal{E}_{i_1} \cap \mathcal{E}_{i_2} \cap \cdots \cap \mathcal{E}_{i_j}].$$

4. Suppose we flip a coin n times to obtain a sequence of flips $X_1, \ldots, X_n$. A streak of flips is a consecutive subsequence of flips that are all the same. For example, if X_3, X_4, and X_5 are all heads, there is a streak of length 3 starting at the third flip. (NB: if X_6 is also heads, then there is also a streak of length 4 starting at the third flip.) Let n be a power of 2. Show that the expected number of streaks of length $\log_2 n + 1$ is $1 - o(1)$.

5. Let Random(i, n) return a sample of a uniform random choice of an integer in the interval $[i, n]$. Show that, given an array $A[1], \ldots, A[n]$, the following algorithm returns an A whose entries have been uniformly randomly permuted:

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for i ← 1, n do
    r ← Random(i, n)
    Exchange A[i] and A[r]
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6. The n passengers on a flight with n seats have been told their seat numbers. They get on the plane one by one. The first person sits in the wrong seat. Subsequent passengers sit in their assigned seats when possible, or otherwise in a randomly chosen empty seat.

   Argue that $\Omega(n)$ people end up in their assigned seats with probability greater than $\frac{1}{2}$. What is the probability that the last passenger finds their seat free?