1. Gaussian Tail Inequalities

Theorem 1. *Let $g \sim \mathcal{N}(0,1)$. Then for any $t > 0$,*

$$\mathbb{P}[g \geq t] \leq \frac{\mathrm{e}^{-\frac{t^2}{2}}}{t\sqrt{2\pi}},$$

and if $t \geq (2\pi)^{-\frac{1}{2}}$, then

$$\mathbb{P}[g \geq t] \leq \mathrm{e}^{-\frac{t^2}{2}}.$$

From the symmetry of Gaussian r.v.s, viz., the fact that $-g$ and g have the same distribution (check this),

$$\begin{aligned}\mathbb{P}[|g| \geq t] &= \mathbb{P}[g \geq t] + \mathbb{P}[g \leq -t] \\ &= \mathbb{P}[g \geq t] + \mathbb{P}[-g \geq t] \\ &= 2\mathbb{P}[g \geq t] \\ &\leq 2\mathrm{e}^{-\frac{t^2}{2}},\end{aligned}$$

assuming $t \geq (2\pi)^{-\frac{1}{2}}$.

Proof of Theorem 1. Write the upper tail as the integral of the gaussian pdf, and use the fact that $\frac{s}{t} \geq 1$ when $s \geq t$:

$$\begin{aligned}\mathbb{P}[g \geq t] &= \frac{1}{\sqrt{2\pi}} \int_t^\infty \mathrm{e}^{-\frac{s^2}{2}}\, ds = \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{t}{t} \mathrm{e}^{-\frac{s^2}{2}}\, ds \\ &\leq \frac{1}{t\sqrt{2\pi}} \int_t^\infty s\mathrm{e}^{-\frac{s^2}{2}}\, ds = \frac{1}{t\sqrt{2\pi}} \left[-\mathrm{e}^{-\frac{s^2}{2}}\right]_t^\infty \\ &= \frac{1}{t\sqrt{2\pi}} \mathrm{e}^{-\frac{t^2}{2}}.\end{aligned}$$

□

2. CLT implications

The classic CLT says that if X_n is the sum of n i.i.d. random variables with finite mean and bounded variance, then $Z_n := \frac{X_n - \mathbb{E}X_n}{\sqrt{\mathrm{Var}(X_n)}} \to \mathcal{N}(0,1)$ in distribution. This means that the CDF of Z_n converges pointwise to that of a $\mathcal{N}(0,1)$ random variable : for all $t \in \mathbb{R}$

$$\mathbb{P}[Z_n \leq t] = \mathbb{P}\left[\frac{X_n - \mathbb{E}X_n}{\sqrt{\mathrm{Var}(X_n)}} \leq t\right] \to \mathbb{P}[g \leq t],$$

as $n \to \infty$, where $g \sim \mathcal{N}(0,1)$.

Some straight-forward implications:

- By considering the probability of the complements of the events $\{Z_n \leq t\}$ and $\{g \leq t\}$, we see that $\mathbb{P}[Z_n > t] \to \mathbb{P}[g > t]$ for all t.
- Using the fact that $\mathbb{P}[Z_n = t] = \mathbb{P}[Z_n \geq t] - \mathbb{P}[Z_n > t]$ and the fact that the two tails on the right converge to the analogous tails for a $\mathcal{N}(0,1)$ variable, we see that $\mathbb{P}[Z_n = t] \to \mathbb{P}[g = t] = 0$.
- It follows that $\mathbb{P}[Z_n \geq t] = \mathbb{P}[Z_n > t] + \mathbb{P}[Z_n = t] \to \mathbb{P}[g > t]$.
- Similar arguments show $\mathbb{P}[Z_n < t] \to \mathbb{P}[g < t]$.

The takeaway is that *all* the tails of Z_n — with and without equality, upper and lower— converge to those of a $\mathcal{N}(0,1)$ random variable. As an example relevant to question 3(i) on Homework 3, this implies that

$$\mathbb{P}[|Z_n| \geq t] = \mathbb{P}[Z_n \geq t] + \mathbb{P}[Z_n \leq -t] \to \mathbb{P}[g \geq t] + \mathbb{P}[g \leq -t] = \mathbb{P}[|g| \geq t]. \tag{1}$$

You should be able to argue up to (1) using what we learned in class (and your knowledge of limits). In fact, the asymptotic CLT has MUCH stronger implications: it implies that any reasonable statistic of Z_n converges to the corresponding statistic of a $\mathcal{N}(0,1)$ random variable. Formally, one way to state this is that when f is a bounded, continuous function,

$$\mathbb{E}[f(Z_n)] \to \mathbb{E}[f(g)]$$

as $n \to \infty$. This result is part of a famous result known as the Portmanteau Theorem that characterizes convergence in distribution. The dominated convergence theorem then implies that such convergence holds for a very large class of functions f, including many that are not continuous. As an example, we can use this result to obtain (1) with much less bean-counting: let $f(z) = \mathbb{1}_{|z|\geq t}(z)$, and observe[1] that

$$\mathbb{P}[|Z_n| \geq t] = \mathbb{E}\mathbb{1}_{|z|\geq t}(Z_n) \to \mathbb{E}\mathbb{1}_{|z|\geq t}(g) = \mathbb{P}[|g| \geq t].$$

Just as Berry–Esseen theorems quantify the rate of convergence of the CDFs, there are versions of the CLT that quantify the rate of convergence of statistics.

[1] As someone pointed out, this isn't quite kosher in the application to Problem 3(i), because the t there is changing with n. In fact, for this value of t, the tail bounds still converge to each other, in the trivial sense that both go to zero.