1. Gaussian Tail Inequalities

**Theorem 1.** Let \( g \sim \mathcal{N}(0, 1) \). Then for any \( t > 0 \),
\[
P[g \geq t] \leq \frac{e^{-t^2/2}}{t\sqrt{2\pi}},
\]
and if \( t \geq (2\pi)^{-1/2} \), then
\[
P[g \geq t] \leq e^{-t^2/2}.
\]

From the symmetry of Gaussian r.v.s, viz., the fact that \(-g\) and \(g\) have the same distribution (check this),
\[
P[|g| \geq t] = P[g \geq t] + P[g \leq -t]
= P[g \geq t] + P[-g \geq t]
= 2P[g \geq t]
\leq 2e^{-t^2/2},
\]
assuming \( t \geq (2\pi)^{-1/2} \).

**Proof of Theorem 1.** Write the upper tail as the integral of the gaussian pdf, and use the fact that \( s^2 \geq 1 \) when \( s \geq t \):
\[
P[g \geq t] = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-s^2/2} \, ds = \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{1}{t} e^{-\frac{s^2}{t^2}} \, ds
\leq \frac{1}{t\sqrt{2\pi}} \int_t^\infty se^{-s^2/2} \, ds
= \frac{1}{t\sqrt{2\pi}} \left[-e^{-s^2/2}\right]_t^\infty
= \frac{1}{t\sqrt{2\pi}} e^{-t^2/2}.
\]
\[\square\]

2. CLT Implications

The classic CLT says that if \( X_n \) is the sum of \( n \) i.i.d. random variables with finite mean and bounded variance, then \( Z_n := \frac{X_n - \mathbb{E}X_n}{\sqrt{\text{Var}(X_n)}} \to \mathcal{N}(0, 1) \) in distribution.

This means that the CDF of \( Z_n \) converges pointwise to that of a \( \mathcal{N}(0, 1) \) random variable : for all \( t \in \mathbb{R} \)
\[
P[Z_n \leq t] = \mathbb{P} \left[ \frac{X_n - \mathbb{E}X_n}{\sqrt{\text{Var}(X_n)}} \leq t \right] \to \mathbb{P}[g \leq t],
\]
as \( n \to \infty \), where \( g \sim \mathcal{N}(0, 1) \).

Some straightforward implications:

- By considering the probability of the complements of the events \( \{Z_n \leq t\}\) and \( \{g \leq t\} \), we see that \( \mathbb{P}[Z_n > t] \to \mathbb{P}[g > t] \) for all \( t \).
- Using the fact that \( \mathbb{P}[Z_n = t] = \mathbb{P}[Z_n \geq t] - \mathbb{P}[Z_n > t] \) and the fact that the two tails on the right converge to the analogous tails for a \( \mathcal{N}(0, 1) \) variable, we see that \( \mathbb{P}[Z_n = t] \to \mathbb{P}[g = t] = 0 \).
- It follows that \( \mathbb{P}[Z_n \geq t] = \mathbb{P}[Z_n > t] + \mathbb{P}[Z_n = t] \to \mathbb{P}[g > t] \).
- Similar arguments show \( \mathbb{P}[Z_n < t] \to \mathbb{P}[g < t] \).
The takeaway is that all the tails of $Z_n$ — with and without equality, upper and lower— converge to those of a $\mathcal{N}(0,1)$ random variable. As an example relevant to question 3(i) on Homework 3, this implies that

(1) $P[|Z_n| \geq t] = P[Z_n \geq t] + P[Z_n \leq -t] \rightarrow P[g \geq t] + P[g \leq -t] = P[|g| \geq t].$

You should be able to argue up to (1) using what we learned in class (and your knowledge of limits). In fact, the asymptotic CLT has MUCH stronger implications: it implies that any reasonable statistic of $Z_n$ converges to the corresponding statistic of a $\mathcal{N}(0,1)$ random variable. Formally, one way to state this is that when $f$ is a bounded, continuous function,

$$E[f(Z_n)] \rightarrow E[f(g)]$$

as $n \rightarrow \infty$. This result is part of a famous result known as the Portmanteau Theorem that characterizes convergence in distribution. The dominated convergence theorem then implies that such convergence holds for a very large class of functions $f$, including many that are not continuous. As an example, we can use this result to obtain (1) with much less bean-counting: let $f(z) = I_{|z| \geq t}(z)$, and observe\footnote{As someone pointed out, this isn’t quite kosher in the application to Problem 3(i), because the $t$ there is changing with $n$. In fact, for this value of $t$, the tail bounds still converge to each other, in the trivial sense that both go to zero.} that

$$P[|Z_n| \geq t] = E[I_{|z| \geq t}(Z_n)] \rightarrow E[I_{|z| \geq t}(g)] = P[|g| \geq t].$$

Just as Berry–Esseen theorems quantify the rate of convergence of the CDFs, there are versions of the CLT that quantify the rate of convergence of statistics.