Let $S = \{x_1 = (k_1, p_1), \ldots, x_n = (k_n, p_n)\}$ be a set of items (keys and priorities) used to construct a max-treap $T(S)$. The p_i are independent $U[0,1]$ random variates, and we assume the keys are unique and sort the items in decreasing order of keys, writing $x_1 > \ldots > x_n$.

1. Outline

To bound the expected height of a treap, we will show a high probability bound on the depth of *any* element in the treap, use a union bound to convert this into a high probability bound on the maximum depth of any element in the treap, then argue this impliest the expected height is small.

2. High probability bound on height of a treap

For the first step of establishing a high probability bound on the depth of an element x_i in the treap, we recall that

$$\text{depth}(x_i) = \sum_{j=1}^{i-1} \xi_j + \sum_{j=i+1}^{n} \xi_j, \tag{1}$$

where $\xi_j = \mathbb{1}_{x_j \text{ is an ancestor of } x_i}$. Recall that x_j is an ancestor of x_i if and only if p_j is the largest priority in the ordered set of items $(x_j, \ldots, x_i)$. Thus knowledge of ξ_j is equivalent to knowing whether or not p_j is the largest priority in the ordered set of items $(x_j, \ldots, x_i)$.

Note that

$$\begin{aligned}\mathbb{P}\left[\bigcap_{j=1}^{i-1} \{\xi_j = \nu_j\}\right] &= \prod_{j=1}^{i-1} \mathbb{P}\big[\xi_j = \nu_j \,\big|\, \xi_{j+1} = \nu_{j+1}, \ldots, \xi_{i-1} = \nu_{i-1}\big] \\ &= \prod_{j=1}^{i-1} \mathbb{P}[\xi_j = \nu_j],\end{aligned}$$

because knowing the *index* of the largest priorities in the subset $(p_{j+1}, \ldots, p_{i-1})$ gives no information about whether or not p_j is larger than $\max(p_{j+1}, \ldots, p_{i-1})$. Thus the Poisson trials in the first sum on the right-hand side of (1) are independent. A similar argument shows the Poisson trials in the second sum are independent [1].

Fix a constant $C > 6$. If the depth of an element is d, then at least one of the sums on the righthand-side of (1) must be greater than $d/2$, so

$$\begin{aligned}\mathbb{P}\left[\text{depth}(x_i) \geq 2C\mathbb{E}[\text{depth}(x_i)]\right] &\leq \mathbb{P}\left[\sum_{j=1}^{i-1} \xi_j \geq C\mathbb{E}[\text{depth}(x_i)]\right] + \mathbb{P}\left[\sum_{j=i+1}^{n} \xi_j \geq C\mathbb{E}[\text{depth}(x_i)]\right] \\ &\leq 2^{1 - C\mathbb{E}[\text{depth}(x_i)]} \end{aligned} \tag{2}$$

where the final inequality follows from a Chernoff bound for Poisson trials and the fact that the expectations of the sums of Poisson trials are both smaller than the expected depth. Now we use the fact[2] that $H_n \geq \ln(n+1)$ to bound the expected depth of x_i below:

$$\begin{aligned}\mathbb{E}[\text{depth}(x_i)] &= H_{n-i+1} + H_i - 2 \\ &\geq \ln(n - i + 2) + \ln(i + 1) \\ &\geq \max(\ln(n - i), \ln(i + 1)) \\ &\geq \ln\left(\frac{n}{2}\right) \\ &= \ln(2)(\log_2(n) - 1)\end{aligned}$$

[1] However, the two sets of summands are not independent of each other. One way to see this is to note that if the first sum is zero, then we know that p_i is the maximum of i independent random $U[0,1]$ variates, so conditioning on that, when $j > i$, by asking if p_j is the largest of $(p_i, p_{i+1}, \ldots, p_j)$, we are essentially asking that p_j not just be the maximum of $j - i + 1$ independent copies of $U[0,1]$, but that it is the maximum of j independent copies. This example shows that, in general, the distribution of ξ_j conditioned on knowledge of the first sum changes from the unconditioned distribution.

[2] This can be shown by comparing H_n to an integral

and recall our previous estimate

$$\mathbb{E}[\text{depth}(x_i)] \leq 2\ln(n).$$

Using these two estimates in (2) gives

$$\begin{aligned}
\mathbb{P}[\text{depth}(x_i) \geq 4C\ln(n)] &\leq \mathbb{P}[\text{depth}(x_i) \geq 2C\mathbb{E}[\text{depth}(x_i)]] \\
&\leq 2^{1-C\mathbb{E}[\text{depth}(x_i)]} \\
&\leq 2^{1+C\ln(2)-C\ln(2)\log_2(n)} \qquad (3) \\
&\leq 2^{1+C}\frac{1}{n^{C/2}}
\end{aligned}$$

Now we can take $C = 6$ and conclude that

$$\mathbb{P}[\text{depth}(x_i) \geq 24\ln(n)] \leq \frac{128}{n^3},$$

which is meaningful when $n \geq 6$. These constants are not optimal— you can be more careful and get tighter bounds— but we only care about asymptotics here, so this is good enough.

A union bound shows

$$\mathbb{P}[\max_{i=1,\ldots,n} \text{depth}(x_i) \geq 24\ln(n)] \leq \frac{128}{n^2}, \qquad (4)$$

which is meaningful when $n \geq 12$.

3. Expected height of a treap

The fact that the expected height of a treap is $O(\ln(n))$ follows immediately from (4). Let $H = \max_{i=1\ldots,n} \text{depth}(x_i)$ be the height of the treap, then

$$\begin{aligned}
\mathbb{E}[H] &= \sum_{\ell=1}^{24\ln(n)} \ell\,\mathbb{P}[H=\ell] + \sum_{\ell=24\ln(n)+1}^{n-1} \ell\,\mathbb{P}[H=\ell] \\
&\leq 24\ln(n)\mathbb{P}[H \leq 24\ln(n)] + (n-1)\mathbb{P}[H \geq 24\ln(n)] \\
&\leq 24\ln(n) + \frac{128}{n} = O(\ln(n)).
\end{aligned}$$