Linear Algebra

Concepts: SVD, EVD, QR decompositions; Rank, Nullity; Positivity; Orthonormality; Vector norms; Frobenius norm; Spectral norm

1. What is the tightest upper bound on $|x^T y|$ in terms of the Euclidean norms of $x$ and $y$?

2. Let matrices $A$ and $B$ have the same dimensions; show that the trace of the matrix $AB^T$ is the same as the inner product of the vectors $a$ and $b$ obtained by stacking the columns of $A$ and $B$.

3. Why is it that $A^T A + I$ is invertible for any matrix $A$?

4. Express $\|A^{-1}\|_2$ in terms of the singular values of $A$.

5. If $A = QR$ is a QR decomposition, give an expression for $A^T A$ in terms of $R$.

6. If $A = USV^T$ is the full SVD of $A$, then what are the full SVDs of $A^T A$ and $AA^T$?
7. If $A = USV^T$ is the full SVD of $A$, how can you read off the rank and nullity of $A$ from just $S$?

8. If $x^T A^T A x = 0$, what can we say about $x$?

9. If $A$ is symmetric (not necessarily positive-definite), how are its eigenvalue decomposition and singular value decomposition related?

10. If $U$ is a matrix with orthonormal columns, argue that $\|Ux\|_2^2 = \|x\|_2^2$.

11. Express the Frobenius norm of $A$ in terms of the trace of the matrix $A^T A$ and argue that it is smaller than the spectral norm of $A$.

12. If $A$ and $B$ are matrices whose columns are respectively $\{a_i\}_i$ and $\{b_i\}_i$, show that $AB^T = \sum_i a_i b_i^T$.

13. Express $\|aa^T\|_F^2$ in terms of the Euclidean length of $a$. 

Probability

Concepts: Independence; Variance; Expectation; Total Probability; Gaussians

1. Let \( p(x, y) \) be the joint pdf for two random variables \( X, Y \) in \( \mathbb{R}^2 \); give expressions for \( p_1(x) \) and \( p_2(y) \), the marginals of \( X \) and \( Y \).

2. If \( X \) and \( Y \) are independent random variables, how is \( p \) related to \( p_1 \) and \( p_2 \)?

3. If \( X \) and \( Y \) are independent, give an expression for the expectation of \( f(x)g(y) \), where \( f \) and \( g \) are arbitrary functions (this expression should not be true in general if \( X \) and \( Y \) are not independent).

4. What can be said about \( P\{E_1 \cap E_2\} \) if the events \( E_1 \) and \( E_2 \) are independent?

5. If \( X, Y, Z \) are independent, what can we say about \( \mathbb{E}(X+Y+Z) \)? If they are not independent?

6. Let \( [\cdot] \) be the indicator function that returns 1 if the argument is true and 0 otherwise; give a simple expression for \( \mathbb{E}([X \in A]) \).

7. Use the law of total probability to argue that \( P\{E_1\} \leq P\{E_1 \cap E_2\} + P\{E_2^c\} \).
8. Give a sufficient condition for when the variance of $X + Y$ equals the sum of the variances of $X$ and $Y$. Provide a proof of your claim.

9. Let $\mathbf{x}$ be a standard multivariate Gaussian in $\mathbb{R}^n$, $\mathbf{O}$ be an orthonormal matrix, and $\mathcal{A}$ be a subset of $\mathbb{R}^n$. Argue that $P\{\mathbf{Ox} \in \mathcal{A}\} = P\{\mathbf{x} \in \mathbf{O}^T \mathcal{A}\}$, and evaluate the integral for the latter probability to argue that $\mathbf{Ox}$ is also a standard multivariate Gaussian.

10. If $X_1, \ldots, X_n$ are independent identically distributed samples drawn from a distribution with finite variance, what can we say about the distribution of their average $\frac{1}{n} \sum_{i=1}^{n} X_i$?