SVD review

Let $A$ be a rank-$\rho$ matrix in $\mathbb{R}^{m \times n}$ with $m \geq n$. Recall that the full SVD of $A$ takes the form $A = U \Sigma V^T$, where $U$ is an $m \times m$ orthonormal matrix (i.e., the columns of $U$ have unit length and are mutually orthogonal; more concisely, $U^T U = I_m$), $V$ is an $n \times n$ orthonormal matrix, and $\Sigma$ is an $m \times n$ diagonal matrix that has nonnegative entries. The columns of $U$ and $V$ are called, respectively, the left and right singular vectors of $A$, and the diagonal entries of $\Sigma$ are called the singular values of $A$. In particular, $A$ has $m$ left singular vectors and $n$ singular values and right singular vectors.

![Diagram of SVD decomposition]

We decompose $U$ as

$$U = \begin{bmatrix} u_1 & \ldots & u_m \end{bmatrix} = \begin{bmatrix} U_{\rho} & U_{\rho}^\perp \end{bmatrix},$$

so that $u_i$ denotes the $i$th left singular vector of $A$, and the first $\rho$ left singular vectors of $A$ constitute the matrix $U_{\rho}$, while the remaining left singular vectors constitute $U_{\rho}^\perp$. Note that $U_{\rho}^T U_{\rho}^\perp = 0$. We similarly decompose the matrix of right singular vectors as

$$V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} = \begin{bmatrix} V_{\rho} & V_{\rho}^\perp \end{bmatrix},$$

and the matrix of singular values as

$$\Sigma = \begin{bmatrix} \Sigma_{\rho} & 0_{m-\rho \times n} \\ 0_{m-\rho \times n} & 0_{n-\rho \times n} \end{bmatrix}.$$

Using this notation, the full SVD of $A$ has the decomposition

$$A = \begin{bmatrix} U_{\rho} & U_{\rho}^\perp \end{bmatrix} \begin{bmatrix} \Sigma_{\rho} & 0_{m-\rho \times n} \\ 0_{m-\rho \times n} & 0_{n-\rho \times n} \end{bmatrix} \begin{bmatrix} V_{\rho}^T \\ (V_{\rho}^\perp)^T \end{bmatrix}. \quad (1)$$

The full SVD is useful because in the decomposition $A = U \Sigma V^T$, the matrices $U$ and $V$ are orthonormal, so are invertible, and preserve Euclidean norms of vectors. It also lets you immediately read off orthogonal bases for the four fundamental subspaces associated with $A$: the kernel/null space (has basis $V_{\rho}^\perp$), the column space (has basis $U_{\rho}$), the row space (has basis $V_{\rho}$), and the cokernel (i.e. the set of vectors so that $x^T A = 0$, equivalently the kernel of $A^T$; this has basis $U_{\rho}^\perp$).

However, as you can check by multiplying out equation (1), we can also write $A = U_{\rho} \Sigma_{\rho} V_{\rho}^T$. This is called the reduced SVD, and is a more condensed factorization that is very useful in practice. Now $U_{\rho}$ and $V_{\rho}$ only contain the singular vectors corresponding to the nonzero singular values of $A$. Note that if $A$ is an invertible matrix then the reduced SVD and full SVD are identical.