Making Precise Statements

Propositions
Compound Propositions and Truth Tables
Predicates and Quantifiers

Last Time

- Sets, \{3, 5, 11\}
- Sequences, 100111001
- Graphs,
- Examples of basic proofs:
  - In 4 rounds of group dating, no one meets more than 12 people.
  - \( x^2 \) is even "is the same as" \( x \) is even.
  - In any group of 6 people there is an orgy of 3 mutual friends or a war of 3 mutual enemies.
  - **Axiom:** The Well Ordering Principle
  - \( \sqrt{2} \) is not rational.

Today: Making Precise Statements

- Making a precise statement: the proposition
- Complicated precise statements: the compound proposition
  - Truth tables
- Claims about many things
  - Predicates
  - Quantifiers
  - Proofs with quantifiers

Statements can be Ambiguous

- \( 2+2=4. \) **T**
- \( 2+2=5. \) **F**
- You may have cake or ice-cream. *(Can you have both?)*
- **IF** pigs can fly **THEN** you get an A. *(Pigs can’t fly. So, can you get an A?)*
- **EVERY** person has a soul mate.
  - There is a single soul mate that **EVERY** person shares.
  - **EVERY** person has their own special soul mate.

Why is ambiguity bad? **Proof!**

We asked questions of our friends to prove 5(b).

**Pop Quiz:** How to prove 5(a)?

A says Sue’s their soul mate;
B says Joe’s their soul mate;
C says Sue’s their soul mate;
D’s soul mate is a red Porsche;
E says Sue’s their soul mate;
F says Sam’s their soul mate.
Propositions are T or F

We use the letters \(p, q, r, s, \ldots\) to represent propositions.

\[ p: \text{Porky the pig can fly.} \quad F \]
\[ q: \text{You got an A.} \quad T? \]
\[ r: \text{Kilam is an American.} \quad T? \]
\[ s: 4^2 \text{ is even.} \quad T \]

To get complex statements, combine basic propositions using logical connectors.

Negation (NOT), \(\neg p\)

The negation \(\neg p\) is T when \(p\) is F, and the negation \(\neg p\) is F when \(p\) is T.

"Porky the pig can fly" is F

So,

\text{IT IS NOT THE CASE THAT} (Porky the pig can fly) is T

Compound Propositions

\[ p: \text{Porky the pig can fly.} \quad F \]
\[ q: \text{You got an A.} \quad T? \]
\[ r: \text{Kilam is an American.} \quad T? \]
\[ s: 4^2 \text{ is even.} \quad T \]

<table>
<thead>
<tr>
<th>Connector</th>
<th>Symbol</th>
<th>An example in words</th>
</tr>
</thead>
<tbody>
<tr>
<td>NOT</td>
<td>(\neg p)</td>
<td>IT IS NOT THE CASE THAT (Porky the pig can fly)</td>
</tr>
<tr>
<td>AND</td>
<td>(p \land q)</td>
<td>(Porky the pig can fly) AND (You got an A)</td>
</tr>
<tr>
<td>OR</td>
<td>(p \lor q)</td>
<td>(Porky the pig can fly) OR (You got an A)</td>
</tr>
<tr>
<td>IF... THEN...</td>
<td>(p \rightarrow q)</td>
<td>IF (Porky the pig can fly) THEN (You got an A)</td>
</tr>
</tbody>
</table>

Conjunction (AND), \(p \land q\)

Both \(p\) and \(q\) must be T for \(p \land q\) to be T; otherwise \(p \land q\) is F.

"Porky the pig can fly" is F

We don’t know whether “You got an A”.

It does not matter.

\((\text{Porky the pig can fly}) \land (\text{You got an A})\) is F
Disjunction (OR), $p \lor q$

Both $p$ and $q$ must be F for $p \lor q$ to be F, otherwise $p \lor q$ is T.

"Porky the pig can fly" is F.

We don't know whether "You got an A".

Now it matters.

(Porky the pig can fly) $\lor$ (You got an A) is T or F

(Depends on whether you got an A.)

**Pop Quiz:** "You can have cake" OR "You can have ice-cream." Can you have both?

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**Implication (IF... THEN...), $p \to q$**

**IF** "Porky the pig can fly" **THEN** "You got an A." **(T/F?)**

Suppose T. Since pigs can’t fly, does it mean you can’t get an A?

**IF** "$n^2$ is even" **THEN** "$n$ is even." **(T)**

Suppose $n^2$ is even. Can we conclude $n \neq 5$?

**IF** "it rained last night" **THEN** "the grass is wet." **(T)**

$p :$ it rained last night
$q :$ the grass is wet

$p \to q$

What does it *mean* for this common-sense implication to be true? What can you conclude? Did it rain last night? Is the grass wet?

---

**Adding New Information to a True Implication: $p$ is T**

**IF** "it rained last night" **THEN** "the grass is wet." **p $\to$ q**

$p :$ it rained last night
$q :$ the grass is wet

$p \to q$

Weather report in morning paper: rain last night. **← new information**

**IF** (it rained last night) **THEN** (the grass is wet) **T**

It rained last night (from the weather report) **T**

Is the grass wet? **YES!** $\therefore q$ **T**

For a true implication $p \to q$, when $p$ is T, you can conclude $q$ is T.
Adding New Information to a True Implication: $q$ is $T$

\[
\text{IF "it rained last night" THEN "the grass is wet."} \\
p : \text{it rained last night} \\
q : \text{the grass is wet} \\
p \rightarrow q
\]

While picking up the morning paper, you see wet grass.

\[
\text{IF (it rained last night) THEN (the grass is wet) } T \\
The grass is wet (from walking outside) \quad T \\
\text{Did it rain last night?} \quad \therefore \quad p \quad T \text{ or } F
\]

For a true implication $p \rightarrow q$, when $q$ is $T$, you cannot conclude $p$ is $T$.

Adding New Information to a True Implication: $q$ is $F$

\[
\text{IF "it rained last night" THEN "the grass is wet."} \\
p : \text{it rained last night} \\
q : \text{the grass is wet} \\
p \rightarrow q
\]

While picking up the paper, you see dry grass.

\[
\text{IF (it rained last night) THEN (the grass is wet) } F \\
\text{If grass is wet (from walking outside)} \quad F \\
\text{Did it rain last night?} \quad \therefore \quad p \quad F
\]

For a true implication $p \rightarrow q$, when $q$ is $F$, you can conclude $p$ is $F$.

Adding New Information to a True Implication: $p$ is $F$

\[
\text{IF "it rained last night" THEN "the grass is wet."} \\
p : \text{it rained last night} \\
q : \text{the grass is wet} \\
p \rightarrow q
\]

Weather report in morning paper: no rain last night.

\[
\text{IF (it rained last night) THEN (the grass is wet) } F \\
\text{It rained last night (from the weather report)} \quad F \\
\text{Is the grass wet?} \quad \therefore \quad q \quad T \text{ or } F
\]

For a true implication $p \rightarrow q$, when $p$ is $F$, you cannot conclude $q$ is $F$.

Implication: Inferences When New Information Comes

\[
\text{IF (Porky the pig can fly) THEN (You got an A)} \\
\text{can be } T \text{ or } F \text{ (phew)}
\]

For a true implication $p \rightarrow q$:

When $p$ is $T$, you can conclude that $q$ is $T$.

When $q$ is $T$, you cannot conclude $p$ is $T$.

When $p$ is $F$, you cannot conclude $q$ is $F$.

When $q$ is $F$, you can conclude $p$ is $F$.
Falsifying “IF (it rained last night) THEN (the grass is wet)”

- You are a scientist collecting data to verify that the implication is valid (true).
- One night it rained. That morning the grass was dry.
- What do you think about the implication now?

This is a falsifying scenario.

\[ p \implies q \text{ only when } p \text{ is } T \text{ and } q \text{ is } F. \text{ In all other cases } p \implies q \text{ is } T. \]

Example: IF (you are hungry OR you are thirsty) THEN you visit the cafeteria

\[(p \lor q) \implies r \]

- \( p \) : you are hungry
- \( q \) : you are thirsty
- \( r \) : you visit the cafeteria

\[ (p \lor q) \implies r \]

- You are thirsty: \( q = T \). In both cases \( r = T \). (you visit the cafeteria)
- You did visit the cafeteria: \( r = T \).
- You did not visit the cafeteria: \( r = F \).
- You are neither hungry nor thirsty.

Implication is \textit{Extremely Important}, \( p \implies q \)

All these are \( p \implies q \) (\( p \) = “it rained last night” and \( q \) = “the grass is wet”):

- If it rained last night then the grass is wet. \( \text{ IF } p \text{ THEN } q \)
- It rained last night implies the grass is wet. \( p \implies q \)
- It rained last night only if the grass is wet. \( p \text{ ONLY IF } q \)
- The grass is wet if it rained last night. \( q \text{ IF } p \)
- The grass is wet whenever it rains. \( q \text{ WHENEVER } p \)

Truth Tables:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \neg p )</th>
<th>( p \land q )</th>
<th>( p \lor q )</th>
<th>( p \implies q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>F</td>
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<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

\[ p \implies q \equiv \neg q \implies \neg p \equiv \neg p \lor q \]

Order is very important: \( p \implies q \) and \( q \implies p \) do \textbf{not} mean the same thing.

IF I’m dead, THEN my eyes are closed \textbf{vs.} IF my eyes are closed, THEN I’m dead

Equivalent Compound Statements

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \implies q )</th>
<th>( \neg q \implies \neg p )</th>
<th>( \neg p \lor q )</th>
<th>( q \implies p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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</table>

Pop Quiz 3.5. Compound propositions are used for program control flow, especially \( \text{ IF... THEN... } \)

\[ \text{if}(x > 0 \land (y > 1 \land x < y)) \]
\[ \text{Execute some instructions.} \]
\[ \text{if}(x > 0 \land y > 1) \]
\[ \text{Execute some instructions.} \]

Use truth-tables to show that both do the same thing. Which do you prefer and why?
Proving an Implication: Reasoning Without Facts

**IF** \( (n^2 \text{ is even}) \text{ THEN } (n \text{ is even}) \).

<table>
<thead>
<tr>
<th></th>
<th>( p: n^2 \text{ is even} )</th>
<th>( q: n \text{ is even} )</th>
<th>( p \rightarrow q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>( q )</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

What is \( n \)? How to prove?

We must show that the highlighted row cannot occur.

In this row, \( q \) is F: \( n = 2k + 1 \).

\[ n^2 = (2k + 1)^2 = 2(2k^2 + 2k) + 1 \]

\( p \) cannot be T. This row cannot happen: \( p \rightarrow q \) is always T.

---

**Predicates Are Like Functions**

ALL cars have four wheels

Define predicate \( P(c) \) and its domain

\[
C = \{ c | c \text{ is a car} \}
\]

\[ P(c) = \text{"car } c \text{ has four wheels"} \]

"for all \( c \) in \( C \), the statement \( P(c) \) is true."

\( \forall c \in C : P(c) \).

(\( \forall \) means "for all")

<table>
<thead>
<tr>
<th>Predicate</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(c) ) = &quot;car ( c ) has four wheels&quot;</td>
<td>( f(x) = x^2 )</td>
</tr>
<tr>
<td>( \text{Input parameter } c \in C )</td>
<td>( \text{parameter } z \in \mathbb{R} )</td>
</tr>
<tr>
<td>( \text{Output statement } P(c) )</td>
<td>( \text{value } f(x) )</td>
</tr>
<tr>
<td>( \text{Example } P(\text{Jen's VW}) = \text{&quot;car 'Jen's VW' has four wheels&quot;} )</td>
<td>( f(5) = 25 )</td>
</tr>
<tr>
<td>( \forall x \in \mathbb{R}, f(x) \geq 0 )</td>
<td>( \forall x \in \mathbb{R}, f(x) \geq 0 )</td>
</tr>
<tr>
<td>( \text{Meaning } \forall c \in C, \text{the statement } P(c) \text{ is true.} )</td>
<td>( \forall x \in \mathbb{R}, f(x) \geq 0 )</td>
</tr>
</tbody>
</table>

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**Quantifiers**

'EVEN' person has A soulmate.

Kilam has some gray hair.

Everyone has some gray hair.

Any map can be colored with 4 colors with adjacent countries having different colors.

Every even integer \( n > 2 \) is the sum of 2 primes (Goldbach, 1742).

Socrates broke this fact.

There exists a creature with blue eyes and blonde hair.

All cars have four wheels.

These statements are more complex because of quantifiers:

EVEN; A; SOME; ANY; ALL; THERE EXISTS.

Compare:

My Ford Escort has four wheels;
All cars have four wheels.

---

There EXISTS a Creature with Blue eyes and Blonde Hair

Define predicate \( Q(a) \) and its domain

\[
A = \{ a | a \text{ is a creature} \}
\]

\[ Q(a) = \text{"a has blue eyes and blonde hair"} \]

"there exists \( a \) in \( A \) for which the statement \( Q(a) \) is true."

\[ \exists a \in A : Q(a). \]

(\( \exists \) means "there exists")

\[ G(a) = \text{"a has blue eyes"} \]

\[ H(a) = \text{"a has blonde hair"} \]

\[ \exists a \in A : (G(a) \land H(a)) \]

(When the domain is understood, we don’t need to keep repeating it. We write \( \exists a : Q(a) \), or \( \exists a : (G(a) \land H(a)) \),...
Negating Quantifiers

IT IS NOT THE CASE THAT (There is creature with blue eyes and blonde hair)
Same as: “All creatures don’t have blue eyes and blonde hair”
\[-(\exists a \in A : Q(a)) \iff \forall a \in A : \neg Q(a)\]

IT IS NOT THE CASE THAT (All cars have four wheels)
Same as: “There is a car which does not have four wheels”
\[-(\forall c \in C : P(c)) \iff \exists c \in C : \neg P(c)\]

When you take the negation inside the quantifier and negate the predicate, you must switch quantifiers: \(\forall \rightarrow \exists, \exists \rightarrow \forall\)

Proofs with Quantifiers

Claim 1. \(\forall n > 2: \text{if } n \text{ is even, then } n \text{ is a sum of two primes.}\) *(Goldbach, 1742)*

Claim 2. \(\exists (a, b, c) \in \mathbb{N}^3: a^2 + b^2 = c^2.\) \((a, b, c) \in \mathbb{N}^3 \text{ means triples of natural numbers}\)

Claim 3. \(\neg \exists (a, b, c) \in \mathbb{N}^3: a^3 + b^3 = c^3.\)

Claim 4. \(\forall (a, b, c) \in \mathbb{N}^3: a^3 + b^3 \neq c^3.\)

Think about what it would take to prove these claims.

Every Person Has a Soul Mate

Define domains and a predicate.
\[A = \{a : a \text{ is a person}\}.\]

\[P(a, b) = “\text{Person } a \text{ has as a soul mate person } b.”\]

- There is some special person \(b\) who is a soul mate to every person \(b\).
  \(\exists b : (\forall a : P(a, b)).\)

- For every person \(a\), they have their own personal soul mate \(b\).
  \(\forall a : (\exists b : P(a, b)).\)

When quantifiers are mixed, the order in which they appear is important for the meaning. Order generally cannot be switched.