Foundations of Computer Science
Lecture 5

Induction: Proving “For All . . . ”

Induction: What and Why?
Induction: Good, Bad and Ugly
Induction, Well-Ordering and the Smallest Counter-Example
Proving “IF . . ., THEN . . .”.

Proving “. . .IF AND ONLY IF . . .”.

Proof patterns:

- direct proof;
  
  ★ If \( x, y \in \mathbb{Q} \), then \( x + y \in \mathbb{Q} \).
  
  ★ If \( 4^x - 1 \) is divisible by 3, then \( 4^{x+1} - 1 \) is divisible by 3.

- contraposition;
  
  ★ If \( r \) is irrational, then \( \sqrt{r} \) is irrational.
  
  ★ If \( x^2 \) is even, then \( x \) is even.

- contradiction.
  
  ★ \( \sqrt{2} \) is irrational.
  
  ★ \( a^2 - 4b \neq 2 \).
  
  ★ \( 2\sqrt{n} + 1/\sqrt{n} + 1 \leq 2\sqrt{n} + 1 \).
Today: Induction, Proving “… for all …”

1. What is induction.

2. Why do we need it?


4. Examples.

5. Induction, Well-Ordering and the Smallest Counter-Example.
Dispensing Postage Using 5¢ and 7¢ Stamps

<table>
<thead>
<tr>
<th>19¢</th>
<th>20¢</th>
<th>21¢</th>
<th>22¢</th>
<th>23¢</th>
</tr>
</thead>
<tbody>
<tr>
<td>7,7,5</td>
<td>5,5,5,5</td>
<td>7,7,7</td>
<td>5,5,5,7</td>
<td>?</td>
</tr>
</tbody>
</table>

Perseverance is a virtue when tinkering.

<table>
<thead>
<tr>
<th>19¢</th>
<th>20¢</th>
<th>21¢</th>
<th>22¢</th>
<th>23¢</th>
<th>24¢</th>
<th>25¢</th>
<th>26¢</th>
<th>27¢</th>
<th>28¢</th>
</tr>
</thead>
<tbody>
<tr>
<td>7,7,5</td>
<td>5,5,5,5</td>
<td>7,7,7</td>
<td>5,5,5,7</td>
<td>–</td>
<td>7,7,5,5</td>
<td>5,5,5,5,5</td>
<td>7,7,7,5</td>
<td>5,5,5,5,7</td>
<td>7,7,7,7</td>
</tr>
</tbody>
</table>

Can every postage greater than 23¢ can be dispensed?

Intuitively yes.

**Induction** formalizes that intuition.
Why Do We Need Induction?

<table>
<thead>
<tr>
<th>Predicate</th>
<th>Claim</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $P(n) = \text{&quot;5¢ and 7¢ stamps can make postage n.&quot;}$</td>
<td>$\forall n \geq 24 : P(n)$</td>
</tr>
<tr>
<td>(ii) $P(n) = \text{&quot;} n^2 - n + 41 \text{ is a prime number.&quot;}$</td>
<td>$\forall n \geq 1 : P(n)$</td>
</tr>
<tr>
<td>(iii) $P(n) = \text{&quot;} 4^n - 1 \text{ is divisible by 3.&quot;}$</td>
<td>$\forall n \geq 1 : P(n)$</td>
</tr>
</tbody>
</table>

TINKER!

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>...</th>
<th>40</th>
<th>41</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^2 - n + 41$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$(4^n - 1)/3$</td>
<td>1</td>
<td>5</td>
<td>21</td>
<td>85</td>
<td>341</td>
<td>1365</td>
<td>5461</td>
<td>21845</td>
<td>...</td>
<td>1601</td>
<td>1681</td>
</tr>
</tbody>
</table>

How can we prove something for all $n \geq 1$? Verification takes too long!
Prove for general $n$. Can be tricky.

**Induction.** Systematic.
Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

We proved:

\[
P(n) = "4^n - 1 \text{ is divisible by 3}."
\]

We proved:

\[
\begin{align*}
\text{IF } & 4^n - 1 \text{ is divisible by 3,} \\
\text{THEN } & 4^{n+1} - 1 \text{ is divisible by 3.}
\end{align*}
\]

Proof. We prove the claim using a direct proof.
1. Assume that $P(n)$ is T, that is $4^n - 1$ is divisible by 3.
2. This means that $4^n - 1 = 3k$ for an integer $k$, or that $4^n = 3k + 1$.
3. Observe that $4^{n+1} = 4 \cdot 4^n$, and since $4^n = 3k + 1$, it follows that
   \[
   4^{n+1} = 4 \cdot (3k + 1) = 12k + 4.
   \]
   Therefore $4^{n+1} - 1 = 12k + 3 = 3(4k + 1)$ is a multiple of 3 ($4k + 1$ is an integer).
4. Since $4^{n+1} - 1$ is a multiple of 3, we have shown that $4^{n+1} - 1$ is divisible by 3.
5. Therefore, $P(n + 1)$ is T.

We proved:

\[
P(n) \rightarrow P(n + 1)
\]

What use is this? (Reasoning in the absense of facts.)
$$4^n - 1$$ is Divisible by 3 for $$n \geq 1$$

$$P(n) = \text{“}4^n - 1 \text{ is divisible by 3.”}$$

$$P(n) \rightarrow P(n + 1)$$

**NEW INFORMATION:**

From tinkering we know that $$P(1)$$ is T: $$4^1 - 3 = 3$$

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n - 1) \rightarrow P(n) \rightarrow \cdots \]
By Induction, $4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = "4^n - 1$ is divisible by 3."

1. $P(1)$ is $T. \checkmark$
2. $P(n) \rightarrow P(n + 1)$ is $T. \checkmark$

By induction, $P(n)$ is $T$ for all $n \geq 1$.

$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots$

$P(n)$ form an infinite chain of dominos. Topple the first and they all fall.

Practice. Exercise 5.2.
**Induction Template**

**Induction to prove:** $\forall n \geq 1 : P(n)$.

**Proof.** We use induction to prove $\forall n \geq 1 : P(n)$.

1. Show that $P(1)$ is T. ("simple" verification.)

2. Show $P(n) \rightarrow P(n + 1)$ for $n \geq 1$

   - **Prove the implication** using direct proof or contraposition.
     - **Direct**
       - Assume $P(n)$ is T.
       - (valid derivations)
       - must show for any $n \geq 1$
       - must use $P(n)$ here
       - **Show** $P(n + 1)$ is T.
     - **Contraposition**
       - Assume $P(n + 1)$ is F.
       - (valid derivations)
       - must show for any $n \geq 1$
       - must use $\neg P(n + 1)$ here
       - **Show** $P(n)$ is F.

3. Conclude: by induction, $\forall n \geq 1 : P(n)$.

- Prove the *implication* $P(n) \rightarrow P(n + 1)$ for a *general* $n \geq 1$. (Often direct proof)
  Why is this easier than just proving $P(n)$ for general $n$?
- Assume $P(n)$ is T, and reformulate it mathematically.
- Somewhere in the proof you *must* use $P(n)$ to prove $P(n + 1)$.
- End with a statement that $P(n + 1)$ is T.
Sum of Integers

\[ 1 + 2 + 3 + \cdots + (n - 1) + n = ? \]

The GREAT Gauss (age 8-10):

\[
\begin{align*}
S(n) &= 1 + 2 + \cdots + n \\
S(n) &= n + n - 1 + \cdots + 1 \\
2S(n) &= (n + 1) + (n + 1) + \cdots + (n + 1) \\
&= n \times (n + 1)
\end{align*}
\]

\[
S(n) = 1 + 2 + 3 + \cdots + (n - 1) + n = \frac{1}{2} n(n + 1)
\]
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \)

**Proof.** (By Induction) \( P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).

1: **[Base case]** \( P(1) \) claims that \( 1 = \frac{1}{2} \times 1 \times (1 + 1) \), which is clearly T.

2: **[Induction step]** We show \( P(n) \rightarrow P(n + 1) \) for all \( n \geq 1 \), using a direct proof. Assume (induction hypothesis) \( P(n) \) is T: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).

Show \( P(n + 1) \) is T: \( \sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 1 + 1) \).

\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) \quad \text{[key step]}
\]

\[
= \frac{1}{2}n(n + 1) + (n + 1) \quad \text{[induction hypothesis \( P(n) \)]}
\]

\[
= \frac{1}{2}(n + 1)(n + 2) \quad \text{[algebra]}
\]

\[
= \frac{1}{2}(n + 1)(n + 1 + 1).
\]

This is exactly what was to be shown. So, \( P(n + 1) \) is T.

3: By induction, \( P(n) \) is T for all \( n \geq 1 \).
VER Y BAD! Induction Step

\[ \sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+2) \]

Compare: \[ \sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) \]

\[ \sum_{i=1}^{n+1} i - \sum_{i=1}^{n} i = (n + 1) \]

\[ \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \]

(phew, nothing bad 😊)

To start, you can NEVER assert (as though its true) what you are trying to prove.
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 = ? \]

Replace Gauss with TINKERING: *method of differences*.

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S(n) )</td>
<td>1</td>
<td>5</td>
<td>14</td>
<td>30</td>
<td>55</td>
<td>91</td>
<td>140</td>
</tr>
<tr>
<td>1st difference</td>
<td>( S'(n) )</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
</tr>
<tr>
<td>2nd difference</td>
<td>( S''(n) )</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>3rd difference</td>
<td>( S'''(n) )</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3’rd difference constant is like 3’rd derivative constant. So guess:

\[ S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3. \]

\[ \begin{array}{c|cccccccccc}
  n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 & 1 & 5 & 14 & 30 & 55 & 91 & 140 & 204 & 285 & 385 \\
\end{array} \]

\[ \begin{array}{ccccccccc}
  a_0 + a_1 + a_2 + a_3 = 1 \\
  a_0 + 2a_1 + 4a_2 + 8a_3 = 5 \\
  a_0 + 3a_1 + 9a_2 + 27a_3 = 14 \\
  a_0 + 4a_1 + 16a_2 + 64a_3 = 30 \\
  a_0 = 0, \ a_1 = \frac{1}{6}, \ a_2 = \frac{1}{2}, \ a_3 = \frac{1}{3} \\
\end{array} \]
Proof: \( S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n+1)(2n+1) \)

Proof. (By induction.) \( P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \).

1. **[Base case]** \( P(1) \), claims that \( 1 = \frac{1}{6} \times 1 \times 2 \times 3 \), which is clearly T.

2. **[Induction step]** Show \( P(n) \rightarrow P(n+1) \) for all \( n \geq 1 \). Direct proof.
   
   Assume (induction hypothesis) \( P(n) \) is T: \( \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \).
   
   Show \( P(n+1) \) is T: \( \sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+2)(2n+3) \).

\[
\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 \quad \text{[key step]}
\]
\[
= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad \text{[induction hypothesis } P(n) ]
\]
\[
= \frac{1}{6}(n+1)(n+2)(2n+3) \quad \text{[algebra]}
\]

This is exactly what was to be shown. So, \( P(n+1) \) is T.

3. By induction, \( P(n) \) is T for all \( n \geq 1 \).

\[\blacksquare\]
Induction Gone Wrong

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow P(6) \rightarrow P(7) \rightarrow \cdots \]

No Base Case.

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \]
False: \( P(n) : n \leq n + 1 \) for all \( n \geq 1. \)

\[ n \leq n + 1 \rightarrow n + 1 \leq n + 2 \quad \text{therefore} \quad P(n) \rightarrow P(n + 1). \]

[Every link is proved, but without the base case, you have \textit{nothing}.

Broken Chain.

\[ P(1) \quad P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \]
False: \( P(n) : \) “all balls in any set of \( n \) balls are the same color.”

\textbf{Induction step.} Suppose any set of \( n \) balls have the same color. Consider any set of \( n + 1 \) balls \( b_1, b_2, \ldots, b_n, b_{n+1} \). So, \( b_1, b_2, \ldots, b_n \) have the same color and \( b_2, b_3, \ldots, b_{n+1} \) have the same color. Thus \( b_1, b_2, b_3, \ldots, b_{n+1} \) have the same color.

\[ P(n) \rightarrow P(n + 1)? \]

[A \textit{single} broken link kills the entire proof.]
Well Ordering Principle

Any non-empty set of natural numbers has a minimum element.

Induction follows from well ordering. Let $P(1)$ and $P(n) \rightarrow P(n + 1)$ be T.

Suppose $P(n_*)$ fails for the smallest counter-example $n_*$ (well-ordering).

$$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n_* - 1) \rightarrow P(n_*) \rightarrow \cdots$$

Now how can $P(n_* - 1) \rightarrow P(n_*)$ be T?

Any induction proof can also be done using well-ordering.
Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

**Proof.** [Induction] \( P(n) : n < 2^n \).

**Base case.** \( P(1) \) is T because \( 1 < 2^1 \).

**Induction.** Assume \( P(n) \) is T: \( n < 2^n \) and show \( P(n + 1) \) is T: \( n + 1 < 2^{n+1} \).

\[
    n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
\]

Therefore \( P(n + 1) \) is T and, by induction, \( P(n) \) is T for \( n \geq 1 \).

**Proof.** [Well-ordering] Proof by **contradiction**.

Assume that there is an \( n \geq 1 \) for which \( n \geq 2^n \).

Let \( n_* \) be the **minimum** such **counter-example**, \( n_* \geq 2^{n_*} \).

Since \( 1 < 2^1 \), \( n_* \geq 2 \). Since \( n_* \geq 2 \), \( \frac{1}{2} n_* \geq 1 \) and so,

\[
    n_* - 1 \geq n_* - \frac{1}{2} n_* = \frac{1}{2} n_* \geq \frac{1}{2} \times 2^{n_*} = 2^{n_*-1}.
\]

So, \( n_* - 1 \) is a **smaller** counter example. **FISHY!**

**The method of minimum counter-example** is very powerful.
Challenge. A circle has $2n$ distinct points, $n$ are red and $n$ are blue. Prove that one can start at a blue point and move clockwise always having passed as many blue points as red.

Practice. All exercises and pop-quizzes in chapter 5.
Strengthen. Problems in chapter 5.