Foundations of Computer Science
Lecture 5

Induction: Proving “For All . . . ”

Induction: What and Why?
Induction: Good, Bad and Ugly
Induction, Well-Ordering and the Smallest Counter-Example
Last Time

Proving “IF . . . , THEN . . . ”.

Proving “. . . IF AND ONLY IF . . . ”.

Proof patterns:
- direct proof;
  - If \( x, y \in \mathbb{Q} \), then \( x + y \in \mathbb{Q} \).
  - If \( 4^x - 1 \) is divisible by 3, then \( 4^{x+1} - 1 \) is divisible by 3.

- contraposition;
  - If \( r \) is irrational, then \( \sqrt{r} \) is irrational.
  - If \( x^2 \) is even, then \( x \) is even.

- contradiction.
  - \( \sqrt{2} \) is irrational.
  - \( a^2 - 4b \neq 2 \).
  - \( 2\sqrt{n + 1}/\sqrt{n + 1} \leq 2\sqrt{n + 1} \).
Today: Induction, Proving “...for all...”

1. What is induction.

2. Why do we need it?


4. Examples.

5. Induction, Well-Ordering and the Smallest Counter-Example.
Dispensing Postage Using 5¢ and 7¢ Stamps

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Can every postage greater than 23¢ can be dispensed?
Dispensing Postage Using 5¢ and 7¢ Stamps

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Can every postage greater than 23¢ can be dispensed?

Intuitively yes.

**Induction** formalizes that intuition.
### Why Do We Need Induction?

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TINKER!

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<td>5</td>
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Claim

\( \forall n \geq 24 : P(n) \)

\( \forall n \geq 1 : P(n) \)

\( \forall n \geq 1 : P(n) \)

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\[ n \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad \cdots \quad 40 \quad 41 \]

\[ n^2 - n + 41 \quad 41✓ \quad 43✓ \quad 47✓ \quad 53✓ \quad 61✓ \quad 71✓ \quad 83✓ \quad 97✓ \cdots \quad 1601✓ \quad 1681✗ \]

\[ (4^n - 1)/3 \quad 1 \quad 5 \quad 21 \quad 85 \quad 341 \quad 1365 \quad 5461 \quad 21845 \cdots \]

How can we prove something for all \( n \geq 1\)? Verification takes too long!
Why Do We Need Induction?

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How can we prove something for \textit{all} \( n \geq 1 \)? Verification takes too long!

Prove for general \( n \). Can be tricky.
Why Do We Need Induction?

Predicate  

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\[ P(n) = n \geq 24 : P(n) \]

(ii) \( P(n) = \text{“} n^2 - n + 41 \text{ a prime number.”} \)  
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How can we prove something for \( \text{all } n \geq 1 \)? Verification takes too long!  
Prove for general \( n \). Can be tricky.  

**Induction.** Systematic.
Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

$P(n) =$ “$4^n - 1$ is divisible by 3.”
Is \( 4^n - 1 \) Divisible by 3 for \( n \geq 1 \)?

\[
P(n) = \text{“} 4^n - 1 \text{ is divisible by 3.} \]

We proved:

\[
\begin{align*}
\text{IF } &4^n - 1 \text{ is divisible by 3,} & \text{THEN } &4^{n+1} - 1 \text{ is divisible by 3.} \\
P(n) & & &P(n+1)
\end{align*}
\]

Proof. We prove the claim using a direct proof.

1: Assume that \( P(n) \) is T, that is \( 4^n - 1 \) is divisible by 3.
2: This means that \( 4^n - 1 = 3k \) for an integer \( k \), or that \( 4^n = 3k + 1 \).
3: Observe that \( 4^{n+1} = 4 \cdot 4^n \), and since \( 4^n = 3k + 1 \), it follows that
   \[
   4^{n+1} = 4 \cdot (3k + 1) = 12k + 4.
   \]
   Therefore \( 4^{n+1} - 1 = 12k + 3 = 3(4k + 1) \) is a multiple of 3 (\( 4k + 1 \) is an integer).
4: Since \( 4^{n+1} - 1 \) is a multiple of 3, we have shown that \( 4^{n+1} - 1 \) is divisible by 3.
5: Therefore, \( P(n + 1) \) is T.
Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

$$P(n) = "4^n - 1 \text{ is divisible by 3.}"

We proved:

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5: Therefore, $P(n + 1)$ is T.

We proved:

$$P(n) \rightarrow P(n + 1)$$
Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

$$P(n) = "4^n - 1 \text{ is divisible by } 3."$$

We proved:

$$\begin{align*}
\text{IF } 4^n - 1 \text{ is divisible by } 3, & \quad \text{THEN } 4^{n+1} - 1 \text{ is divisible by } 3. \\
P(n) & \implies P(n+1)
\end{align*}$$

Proof. We prove the claim using a direct proof.
1: Assume that $P(n)$ is $T$, that is $4^n - 1$ is divisible by 3.
2: This means that $4^n - 1 = 3k$ for an integer $k$, or that $4^n = 3k + 1$.
3: Observe that $4^{n+1} = 4 \cdot 4^n$, and since $4^n = 3k + 1$, it follows that $4^{n+1} = 4 \cdot (3k + 1) = 12k + 4$.
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We proved:

$$P(n) \to P(n + 1)$$

What use is this? (Reasoning in the absense of facts.)
$4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = "4^n - 1$ is divisible by 3.”

$P(n) \rightarrow P(n + 1)$
$4^n - 1$ is Divisible by $3$ for $n \geq 1$

\[ P(n) = \text{“}4^n - 1 \text{ is divisible by } 3\text{.”} \]

\[ P(n) \rightarrow P(n + 1) \]

NEW INFORMATION:
From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3$  \hspace{1cm} \leftarrow \text{divisible by } 3 \text{ (new fact)}

✓ $P(1)$
4^n - 1 is Divisible by 3 for \( n \geq 1 \)

\[ P(n) = "4^n - 1 \text{ is divisible by 3."} \]

\[ P(n) \rightarrow P(n + 1) \]

NEW INFORMATION:

From tinkering we know that \( P(1) \) is T: \( 4^1 - 3 = 3 \)

\( \checkmark\) \( P(1) \rightarrow P(2) \)
$4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = \text{"}4^n - 1 \text{ is divisible by 3."}$

$P(n) \rightarrow P(n + 1)$

**NEW INFORMATION:**
From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3 \leftarrow$ divisible by 3 (new fact)

✓ $P(1) \rightarrow P(2)$
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\[ P(n) = "4^n - 1 \text{ is divisible by 3.}" \]

\[ P(n) \rightarrow P(n + 1) \]

**NEW INFORMATION:**
From tinkering we know that \( P(1) \) is T: \( 4^1 - 3 = 3 \) ← divisible by 3 (new fact)

\[ \checkmark \ P(1) \rightarrow \checkmark \ P(2) \rightarrow P(3) \]
4^n − 1 is Divisible by 3 for n ≥ 1

\[ P(n) = \text{“}4^n − 1 \text{ is divisible by 3.”} \]

\[ P(n) \rightarrow P(n + 1) \]

NEW INFORMATION:
From tinkering we know that \( P(1) \) is T: \( 4^1 − 3 = 3 \)

\[ P(1) \rightarrow P(2) \rightarrow P(3) \]

\( \leftarrow \) divisible by 3 (new fact)
4^n - 1 is Divisible by 3 for \( n \geq 1 \)

\[ P(n) = "4^n - 1 is divisible by 3." \]

\[ P(n) \rightarrow P(n + 1) \]

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NEW INFORMATION:
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\[ \checkmark \ P(1) \rightarrow \checkmark \ P(2) \rightarrow \checkmark \ P(3) \rightarrow \checkmark \ P(4) \rightarrow \cdots \rightarrow \checkmark \ P(n - 1) \]
$4^n - 1$ is Divisible by 3 for $n \geq 1$

\[ P(n) = "4^n - 1 \text{ is divisible by 3.}" \]

\[ P(n) \rightarrow P(n + 1) \]

NEW INFORMATION:
From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3$ \hspace{1cm} $\leftarrow$ divisible by 3 (new fact)

\[ \checkmark P(1) \rightarrow \checkmark P(2) \rightarrow \checkmark P(3) \rightarrow \checkmark P(4) \rightarrow \cdots \rightarrow \checkmark P(n - 1) \rightarrow \checkmark P(n) \]
**4^n − 1 is Divisible by 3 for n ≥ 1**

\[ P(n) = "4^n − 1 is divisible by 3." \]

\[ P(n) \rightarrow P(n + 1) \]

**NEW INFORMATION:**
From tinkering we know that \( P(1) \) is T: \( 4^1 − 3 = 3 \)

\[ \checkmark \rightarrow \checkmark \rightarrow \checkmark \rightarrow \checkmark \rightarrow \cdots \rightarrow \checkmark \rightarrow \checkmark \rightarrow \cdots \]

← divisible by 3 (new fact)
By Induction, $4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = "4^n - 1$ is divisible by 3."

1. $P(1)$ is T. ✓
2. $P(n) \rightarrow P(n + 1)$ is T. ✓

\[ \begin{align*} 
&\text{By induction, } P(n) \text{ is T for all } n \geq 1. 
\end{align*} \]
By Induction, $4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = \text{"}4^n - 1 \text{ is divisible by 3."}$

1. $P(1)$ is T.✓
2. $P(n) \to P(n+1)$ is T.✓

By induction, $P(n)$ is T for all $n \geq 1$.

$P(1) \to P(2) \to P(3) \to P(4) \to P(5) \to \cdots$

$P(n)$ form an infinite chain of dominos.
Topple the first and they *all* fall.

Practice. Exercise 5.2.
**Induction to prove:** \( \forall n \geq 1 : P(n) \).

**Proof.** We use induction to prove \( \forall n \geq 1 : P(n) \).
**Induction to prove:** $\forall n \geq 1 : P(n)$.

*Proof.* We use induction to prove $\forall n \geq 1 : P(n)$.

1. Show that $P(1)$ is T. (“simple” verification.) [base case]
Induction to prove: $\forall n \geq 1 : P(n)$.

*Proof.* We use induction to prove $\forall n \geq 1 : P(n)$.

1. Show that $P(1)$ is T. (“simple” verification.)

2. Show $P(n) \rightarrow P(n + 1)$ for $n \geq 1$
**Induction Template**

**Induction to prove:** \( \forall n \geq 1 : P(n) \).

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1. Show that \( P(1) \) is T. ("simple" verification.)  
   
   **[base case]**

2. Show \( P(n) \rightarrow P(n + 1) \) for \( n \geq 1 \)

   **[induction step]**

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| Assume \( P(n) \) is T.  
| (valid derivations)  
| must show for any \( n \geq 1 \)  
| must use \( P(n) \) here  
| **Show** \( P(n + 1) \) **is T.** |
| **Contraposition**  
| Assume \( \overline{P(n + 1)} \) is F.  
| (valid derivations)  
| must show for any \( n \geq 1 \)  
| must use \( \overline{P(n + 1)} \) here  
| **Show** \( P(n) \) **is F.** |
Induction to prove: \( \forall n \geq 1 : P(n) \).

**Proof.** We use induction to prove \( \forall n \geq 1 : P(n) \).

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| Assume \( \neg P(n + 1) \) is F.
| (valid derivations)
| must show for any \( n \geq 1 \)
| must use \( \neg P(n + 1) \) here
| Show \( P(n) \) **is** F. |

3: Conclude: by induction, \( \forall n \geq 1 : P(n) \).
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**Induction to prove:** $\forall n \geq 1 : P(n)$.

**Proof.** We use induction to prove $\forall n \geq 1 : P(n)$.

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2. Show $P(n) \rightarrow P(n + 1)$ for $n \geq 1$

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   | Contraposition                                              |
   | Assume $P(n + 1)$ is F. (valid derivations)                |
   | must show for any $n \geq 1$                               |
   | must use $\neg P(n + 1)$ here                             |
   | **Show $P(n)$ is F.**                                      |

3. Conclude: by induction, $\forall n \geq 1 : P(n)$.

- Prove the implication $P(n) \rightarrow P(n + 1)$ for a *general* $n \geq 1$. (Often direct proof)

   Why is this easier than just proving $P(n)$ for general $n$?
**Induction to prove:** $\forall n \geq 1 : P(n)$.

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3. Conclude: by induction, $\forall n \geq 1 : P(n)$.  
   [ ]

- Prove the *implication* $P(n) \rightarrow P(n + 1)$ for a *general* $n \geq 1$. (Often direct proof)
  Why is this easier than just proving $P(n)$ for general $n$?

- Assume $P(n)$ is T, and reformulate it mathematically.
**Induction Template**

**Induction to prove:** $\forall n \geq 1 : P(n)$.

**Proof.** We use induction to prove $\forall n \geq 1 : P(n)$.

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   - **[base case]**

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3. Conclude: by induction, $\forall n \geq 1 : P(n)$.

- Prove the *implication* $P(n) \rightarrow P(n + 1)$ for a *general* $n \geq 1$. (Often direct proof)
  Why is this easier than just proving $P(n)$ for general $n$?

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- Somewhere in the proof you must use $P(n)$ to prove $P(n + 1)$.
Induction Template

**Induction to prove:** \( \forall n \geq 1 : P(n) \).

**Proof.** We use induction to prove \( \forall n \geq 1 : P(n) \).

1. Show that \( P(1) \) is T. ("simple" verification.)  
   [base case]

2. Show \( P(n) \rightarrow P(n + 1) \) for \( n \geq 1 \)  
   [induction step]
   
   **Prove the implication** using direct proof or contraposition.

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     Assume \( P(n) \) is T.  
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     must show for any \( n \geq 1 \)  
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     Assume \( \neg P(n + 1) \) is T.  
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     must show for any \( n \geq 1 \)  
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3. Conclude: by induction, \( \forall n \geq 1 : P(n) \).  
   ■

- Prove the *implication* \( P(n) \rightarrow P(n + 1) \) for a *general* \( n \geq 1 \). (Often direct proof)
  
  Why is this easier than just proving \( P(n) \) for general \( n \)?

- Assume \( P(n) \) is T, and reformulate it mathematically.

- Somewhere in the proof you *must* use \( P(n) \) to prove \( P(n + 1) \).

- End with a statement that \( P(n + 1) \) is T.
Sum of Integers

\[ 1 + 2 + 3 + \cdots + (n - 1) + n = ? \]
Sum of Integers

\[ 1 + 2 + 3 + \cdots + (n - 1) + n = ? \]

The GREAT Gauss (age 8-10):

\[
\begin{align*}
S(n) &= 1 + 2 + \cdots + n \\
S(n) &= n + n - 1 + \cdots + 1 \\
2S(n) &= (n + 1) + (n + 1) + \cdots + (n + 1) \\
&= n \times (n + 1)
\end{align*}
\]
Sum of Integers

\[
1 + 2 + 3 + \cdots + (n-1) + n = ?
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The GREAT Gauss (age 8-10):

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2S(n) = (n+1) + (n+1) + \cdots + (n+1)
\]
\[
= n \times (n+1)
\]

\[
S(n) = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)
\]
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \)

Proof. (By Induction) \( P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).
Proof: \[ \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \]

Proof. (By Induction) \( P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).

1: [Base case] \( P(1) \) claims that \( 1 = \frac{1}{2} \times 1 \times (1 + 1) \), which is clearly T.
Proof: $\sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$

Proof. (By Induction) $P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$.

1. [Base case] $P(1)$ claims that $1 = \frac{1}{2} \times 1 \times (1 + 1)$, which is clearly T.

2. [Induction step] We show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$, using a direct proof.
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2} n(n + 1) \)

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   Assume (induction hypothesis) \( P(n) \) is \( T \): \( \sum_{i=1}^{n} i = \frac{1}{2} n(n + 1) \).

   Show \( P(n + 1) \) is \( T \): \( \sum_{i=1}^{n+1} i = \frac{1}{2} (n + 1)(n + 1 + 1) \).
Proof. (By Induction) $P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$.

1: [Base case] $P(1)$ claims that $1 = \frac{1}{2} \times 1 \times (1 + 1)$, which is clearly true.

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Assume (induction hypothesis) $P(n)$ is true: $\sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$.

Show $P(n + 1)$ is true: $\sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 1 + 1)$.

\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) \quad \text{[key step]}
\]
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \)

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\]

\[
= \frac{1}{2}n(n + 1) + (n + 1) \quad \text{[induction hypothesis \( P(n) \)]}
\]
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \)

*Proof. (By Induction) P(n) : \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).*

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= \frac{1}{2}n(n + 1) + (n + 1) \quad \text{[induction hypothesis P(n)]}
\]

\[
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\]

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= \frac{1}{2}(n + 1)(n + 1 + 1).
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\]

This is exactly what was to be shown. So, \( P(n + 1) \) is \( T \).
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \)

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1: [Base case] \( P(1) \) claims that \( 1 = \frac{1}{2} \times 1 \times (1 + 1) \), which is clearly \( \top \).

2: [Induction step] We show \( P(n) \rightarrow P(n + 1) \) for all \( n \geq 1 \), using a direct proof.
   \begin{align*}
   \text{Assume (induction hypothesis) } & P(n) \text{ is } \top : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1). \\
   \text{Show } P(n + 1) \text{ is } & \top : \sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 1 + 1).
   \end{align*}

\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) \quad \text{[key step]}
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\[
= \frac{1}{2}(n + 1)(n + 1 + 1).
\]

This is exactly what was to be shown. So, \( P(n + 1) \) is \( \top \).

3: By induction, \( P(n) \) is \( \top \) for all \( n \geq 1 \). 

\[\blacksquare\]
\[
\sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 2)
\]

(What we want)
**VERY BAD!** Induction Step

\[ \sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 2) \quad \text{(What we want)} \]

\[ \sum_{i=1}^{n+1} i - (n + 1) = \frac{1}{2}(n + 1)(n + 2) - (n + 1) \]
Induction Step

\[ \sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 2) \]  
(What we want)

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\[ \sum_{i=1}^{n} i = \frac{1}{2}(n + 1)(n + 2) - (n + 1) \]

\[ \sum_{i=1}^{n} i = (n + 1)\left(\frac{n}{2} + 1 - 1\right) \]
Induction Step

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\sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 2) \quad \text{(What we want)}
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\[
\sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \checkmark \quad \text{(phew, nothing bad 😊)}
\]
Very Bad! Induction Step

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\sum_{i=1}^{n+1} i = \frac{1}{2} (n + 1)(n + 2) \quad \text{(What we want)}
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\sum_{i=1}^{n} i = \frac{1}{2} n(n + 1) \checkmark \quad \text{(phew, nothing bad 😊)}
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\sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 2) \\
\sum_{i=1}^{n+1} i - (n + 1) = \frac{1}{2}(n + 1)(n + 2) - (n + 1)
\]

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\]

Compare: \[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1)
\]

To start, you can **NEVER** assert (as though its true) what you are trying to prove.
$S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 = ?$

Where’s the GREAT Gauss when you need him?
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 = ? \]

Replace Gauss with TINKERING: *method of differences*.

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Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 = ? \]

Replace Gauss with TINKERING: \textit{method of differences}.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
S(n) & 1 & 5 & 14 & 30 & 55 & 91 & 140 \\
\hline
1st difference S'(n) & 4 & 9 & 16 & 25 & 36 & 49 \\
\hline
\end{array}
\]
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 = ? \]

Replace Gauss with TINKERING: *method of differences.*

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</tr>
<tr>
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3\'rd difference constant is like 3\'rd derivative constant.
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 = ? \]

Replace Gauss with TINKERING: \textit{method of differences}.

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\[ S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3. \]
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\[ S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3. \]

\[
\begin{align*}
a_0 + a_1 + a_2 + a_3 &= 1 \\
a_0 + 2a_1 + 4a_2 + 8a_3 &= 5 \\
a_0 + 3a_1 + 9a_2 + 27a_3 &= 14 \\
a_0 + 4a_1 + 16a_2 + 64a_3 &= 30 \\
\end{align*}
\]

\[ a_0 = 0, \ a_1 = \frac{1}{6}, \ a_2 = \frac{1}{2}, \ a_3 = \frac{1}{3} \]
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = ? \]

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\[ a_0 + \frac{1}{6}, \ a_1 = \frac{1}{6}, \ a_2 = \frac{1}{2}, \ a_3 = \frac{1}{3} \]
Proof: $S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n+1)(2n+1)$

Proof. (By induction.) $P(n): \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$. 
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   This is exactly what was to be shown. So, \( P(n + 1) \) is true.
Proof: \( S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n+1)(2n+1) \)

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1: [Base case] \( P(1) \), claims that \( 1 = \frac{1}{6} \times 1 \times 2 \times 3 \), which is clearly T.

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Assume (induction hypothesis) \( P(n) \) is T: \( \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \).

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3: By induction, \( P(n) \) is T for all \( n \geq 1 \).
Induction Gone Wrong

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow P(6) \rightarrow P(7) \rightarrow \cdots \]
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**No Base Case.**

\[ P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \]

False: \( P(n) : n \leq n + 1 \) for all \( n \geq 1 \).

\[ n \leq n + 1 \rightarrow n + 1 \leq n + 2 \] therefore \( P(n) \rightarrow P(n + 1) \).
Induction Gone Wrong

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Broken Chain.

\[ P(1) \quad P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \]

False: \( P(n) : \) “all balls in any set of \( n \) balls are the same color.”

**Induction step.** Suppose any set of \( n \) balls have the same color. Consider any set of \( n + 1 \) balls \( b_1, b_2, \ldots, b_n, b_{n+1} \). So, \( b_1, b_2, \ldots, b_n \) have the same color and \( b_2, b_3, \ldots, b_{n+1} \) have the same color. Thus \( b_1, b_2, b_3, \ldots, b_{n+1} \) have the same color.

\[ P(n) \rightarrow P(n + 1)? \]
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\[ P(n) \rightarrow P(n + 1)? \]

[A single broken link kills the entire proof.]
Well-ordering Principle.

*Any* non-empty set of natural numbers has a minimum element.
Well Ordering Principle

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Suppose $P(n_*)$ fails for the smallest counter-example $n_*$ (well-ordering).
Well Ordering Principle

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\[
\begin{align*}
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\end{align*}
\]
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\]

Now how can $P(n_* - 1) \to P(n_*)$ be $\top$?
Well Ordering Principle

**Well-ordering Principle.**

*Any* non-empty set of natural numbers has a minimum element.

Induction follows from well ordering. Let $P(1)$ and $P(n) \rightarrow P(n + 1)$ be T.

Suppose $P(n_*)$ fails for the **smallest** counter-example $n_*$ (well-ordering).

\[
P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n_* - 1) \rightarrow P(n_*) \rightarrow \cdots
\]

Now how can $P(n_* - 1) \rightarrow P(n_*)$ be T?

Any induction proof can also be done using well-ordering.
Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

\textit{Proof.} [Induction] \( P(n) : n < 2^n \).
Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

Proof. [Induction] $P(n) : n < 2^n$.

**Base case.** $P(1)$ is T because $1 < 2^1$. 

Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

**Proof.** [Induction] \( P(n) \): \( n < 2^n \).

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**Induction.** Assume \( P(n) \) is T: \( n < 2^n \) and show \( P(n + 1) \) is T: \( n + 1 < 2^{n+1} \).

\[
    n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
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Therefore $P(n + 1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$. \[
\]

Creator: Malik Magdon-Ismail

Induction: Proving “For All…”: 17/18

Getting Good at Induction →
Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

*Proof.* [Induction] $P(n): n < 2^n$.

**Base case.** $P(1)$ is T because $1 < 2^1$.

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\[ n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}. \]

Therefore $P(n + 1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$.

*Proof.* [Well-ordering] Proof by **contradiction**.
Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

**Proof.** [Induction] \( P(n) : n < 2^n \).

**Base case.** \( P(1) \) is \( \text{T} \) because \( 1 < 2^1 \).

**Induction.** Assume \( P(n) \) is \( \text{T} \): \( n < 2^n \). and show \( P(n + 1) \) is \( \text{T} \): \( n + 1 < 2^{n+1} \).

\[
\begin{align*}
  n + 1 & \leq n + n = 2n \\
  & \leq 2 \times 2^n = 2^{n+1}.
\end{align*}
\]

Therefore \( P(n + 1) \) is \( \text{T} \) and, by induction, \( P(n) \) is \( \text{T} \) for \( n \geq 1 \).

**Proof.** [Well-ordering] Proof by **contradiction**.

Assume that there is an \( n \geq 1 \) for which \( n \geq 2^n \).
Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

Proof. [Induction] \( P(n) : n < 2^n \).

**Base case.** \( P(1) \) is \( T \) because \( 1 < 2^1 \).

**Induction.** Assume \( P(n) \) is \( T \): \( n < 2^n \). and show \( P(n+1) \) is \( T \): \( n + 1 < 2^{n+1} \).

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\end{align*}
\]

Therefore \( P(n+1) \) is \( T \) and, by induction, \( P(n) \) is \( T \) for \( n \geq 1 \).


Assume that there is an \( n \geq 1 \) for which \( n \geq 2^n \).

Let \( n^* \) be the minimum such counter-example, \( n^* \geq 2^{n^*} \).
Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

Proof. [Induction] $P(n) : n < 2^n$.

Base case. $P(1)$ is T because $1 < 2^1$.

Induction. Assume $P(n)$ is T: $n < 2^n$. and show $P(n + 1)$ is T: $n + 1 < 2^{n+1}$.

\[
\begin{align*}
n + 1 & \leq n + n = 2n \\
& \leq 2 \times 2^n = 2^{n+1}.
\end{align*}
\]

Therefore $P(n + 1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$.


Assume that there is an $n \geq 1$ for which $n \geq 2^n$.

Let $n_*$ be the minimum such counter-example, $n_* \geq 2^{n_*}$.

Since $1 < 2^1$, $n_* \geq 2$. Since $n_* \geq 2$, $\frac{1}{2}n_* \geq 1$ and so,

\[
n_* - 1 \geq n_* - \frac{1}{2}n_* = \frac{1}{2}n_* \geq \frac{1}{2} \times 2^{n_*} = 2^{n_*-1}.
\]
Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

**Proof.** [Induction] $P(n) : n < 2^n$.

**Base case.** $P(1)$ is T because $1 < 2^1$.

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\[
n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
\]

Therefore $P(n + 1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$.

Proof. [Well-ordering] Proof by **contradiction**.

Assume that there is an $n \geq 1$ for which $n \geq 2^n$.

Let $n_*$ be the **minimum** such **counter-example**, $n_* \geq 2^{n_*}$.

Since $1 < 2^1$, $n_* \geq 2$. Since $n_* \geq 2$, $\frac{1}{2}n_* \geq 1$ and so,

\[
n_* - 1 \geq n_* - \frac{1}{2}n_* = \frac{1}{2}n_* \geq \frac{1}{2} \times 2^{n_*} = 2^{n_*-1}.
\]

So, $n_* - 1$ is a **smaller** counter example. **FISHY!**
Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

Proof. [Induction] \( P(n) : n < 2^n \).

**Base case.** \( P(1) \) is T because \( 1 < 2^1 \).

**Induction.** Assume \( P(n) \) is T: \( n < 2^n \). and show \( P(n + 1) \) is T: \( n + 1 < 2^{n+1} \).

\[
 n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
\]

Therefore \( P(n + 1) \) is T and, by induction, \( P(n) \) is T for \( n \geq 1 \).

Proof. [Well-ordering] Proof by **contradiction**.

Assume that there is an \( n \geq 1 \) for which \( n \geq 2^n \).

Let \( n_* \) be the **minimum** such **counter-example**, \( n_* \geq 2^{n_*} \). \hfill \leftarrow \text{well ordering}

Since \( 1 < 2^1 \), \( n_* \geq 2 \). Since \( n_* \geq 2 \), \( \frac{1}{2} n_* \geq 1 \) and so,

\[
 n_* - 1 \geq n_* - \frac{1}{2} n_* = \frac{1}{2} n_* \geq \frac{1}{2} \times 2^{n_*} = 2^{n_*-1}.
\]

So, \( n_* - 1 \) is a **smaller** counter example. **FISHY!**

The **method of minimum counter-example** is very powerful.
Challenge. A circle has $2n$ distinct points, $n$ are red and $n$ are blue. Prove that one can start at a blue point and move clockwise always having passed as many blue points as red.

Practice. All exercises and pop-quizzes in chapter 5.
Strengthen. Problems in chapter 5.