Foundations of Computer Science
Lecture 6

Strong Induction
Strengthening the Induction Hypothesis
Many Flavors of Induction

Last Time

1. Proving “for all”:
   - \( P(n) : 4n - 1 \) is divisible by 3. \( \forall n : P(n) \)
   - \( P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \). \( \forall n : P(n) \)
   - \( P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{2}n(n + 1)(2n + 1) \). \( \forall n : P(n) \)

2. Induction.
   - Induction and Well-Ordering.

A Hard Problem: \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \)

Proof. \( P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \)

1. [Base case] \( P(1) \) claims that \( 1 \leq 2 \sqrt{1} \), which is clearly true.

2. [Induction step] Show \( P(n) \rightarrow P(n+1) \) for all \( n \geq 1 \) (direct proof).
   - Assume (induction hypothesis) \( P(n) \) is true. \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \).
   - Show \( P(n+1) \) is true: \( \sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1} \).

3. By induction, \( P(n) \) is true for all \( n \geq 1 \).
Proving Stronger Claims

\[ n^2 \leq 2^n \quad \text{for } n \geq 4. \]

**Induction Step.** Must use \( n^2 \leq 2^n \) to show \( (n + 1)^2 \leq 2^{n+1} \).

\[ (n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n + 2^n + 2^n = 2^{n+1} \]

What to do with the \( 2n + 1 \)?

Would be fine if \( 2n + 1 \leq 2^n \).

With induction, it can be easier to prove a stronger claim.

---

**Strengthen the Claim: \( Q(n) \) Implies \( P(n) \)**

\[ Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n. \]

\[ Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots \]

**Proof.** \( Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n. \)

1. **Base case** \( Q(4) \) claims \( (i) \ 4^2 \leq 2^4 \) AND \( (ii) \ 2 \times 4 + 1 \leq 2^4 \). Both clearly T.

2. **Induction step** Show \( Q(n) \rightarrow Q(n + 1) \) for \( n \geq 4 \) (direct proof).

Assume (induction hypothesis) \( Q(n) \) is T: \( (i) \ n^2 \leq 2^n \) AND \( (ii) \ 2n + 1 \leq 2^n \).

Show \( Q(n + 1) \) is T: \( (i) \ (n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \checkmark \) (because from the induction hypothesis \( n^2 \leq 2^n \) and \( 2n + 1 \leq 2^n \)).

\[ (ii) \ 2n + 1 + 1 = 2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \checkmark \] (because \( 2 \leq 2^n \) and from the induction hypothesis \( 2n + 1 \leq 2^n \)).

So, \( Q(n + 1) \) is T.

3. By induction, \( Q(n) \) is T \( \forall n \geq 4 \).

---

**L-Tile Land: Induction Idea**

Suppose \( P(n) \) is T. What about \( P(n + 1) \)?

The \( 2^{n+1} \times 2^{n+1} \) patio can be decomposed into four \( 2^n \times 2^n \) patios.

\[ \begin{array}{cc}
2^n & 2^n \\
2^n & 2^n \\
\end{array} \]

Add first tile in the center. Now each sub-patio has one missing square.

**Problem.** Corner squares are missing. \( P(n) \) can be used only if center-square is missing.

**Solution.** Strengthen claim to also include patios missing corner-squares.

\[ Q(n) : (i) \ The \ 2^n \times 2^n \ grid \ missing \ a \ \text{center-square} \ can \ be \ L-tiled; \quad \text{AND} \quad (ii) \ The \ 2^n \times 2^n \ grid \ missing \ a \ \text{corner-square} \ can \ be \ L-tiled. \]
L-Tile Land: Induction Proof of Stronger Claim

Assume $Q(n)$: (i) The $2^i \times 2^i$ grid missing a center-square can be $L$-tiled; AND (ii) The $2^i \times 2^i$ grid missing a corner-square can be $L$-tiled.

Induction step: Must prove two things for $Q(n + 1)$, namely (i) and (ii).

(i) Center square missing.

(ii) Corner square missing.

Your task: Add base cases and complete the formal proof.

Exercise 6.4. What if the missing square is some random square? Strengthen further.

Leaping Induction

Induction. One base case.

Leaping Induction. More than one base case.

Example. Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

<table>
<thead>
<tr>
<th>3c</th>
<th>4c</th>
<th>5c</th>
<th>6c</th>
<th>7c</th>
<th>8c</th>
<th>9c</th>
<th>10c</th>
<th>11c</th>
<th>12c</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>5</td>
<td>3,3</td>
<td>4,4</td>
<td>3,3</td>
<td>3,4</td>
<td>4,4</td>
<td>4,4</td>
<td>4,4</td>
<td>...</td>
</tr>
</tbody>
</table>

$P(n)$: Postage of $n$ cents can be made using only 3¢ and 4¢ stamps.

$P(n) \rightarrow P(n + 3)$

(add a 3¢ stamp to $n$)

Base cases: 6¢, 7¢, 8¢.

A Tricky Induction Problem

$P(n) : n^3 < 2^n$, for $n \geq 10$. (Exercise 6.2)

Suppose $P(n)$ is T. Consider $P(n + 2) : (n + 2)^3 < 2^{n+2}^2$?

$(n + 2)^3 = n^3 + 6n^2 + 12n + 8$

$\leq n^3 + n \cdot n^2 + n^2 \cdot n + n^3$

$(n \geq 10 \rightarrow 6 < n; 12 < n^2; 8 < n^3)$

$= 4n^3 < 4 \cdot 2^n = 2^n < 2^{n+2}$(P(n) gives $n^3 < 2^n$)

$P(n) \rightarrow P(n + 2)$.

Base cases. $P(10) : 10^3 < 2^{10}\checkmark$ and $P(11) : 11^3 < 2^{11}\checkmark$

Fundamental Theorem of Arithmetic

2015 = 5 × 13 × 31.

(Theorem. (The Primes $P = \{2, 3, 5, 7, 11, \ldots\}$ are the atoms for numbers.)

Suppose $n \geq 2$. Then.
- $n$ can be written as a product of factors all of which are prime.
- The representation of $n$ as a product of primes is unique (up to reordering).

$P(n) : n$ is a product of primes.

What’s the first thing we do? TINKER!

2016 = 2 × 2 × 2 × 2 × 3 × 3 × 7.

Wow! No similarity between the factors of 2015 and those of 2016.

How will $P(n)$ help us to prove $P(n + 1)$?
Much “Stronger” Induction Claim

Do smaller values of $n$ help with 2016? Yes!

$2016 = 32 \times 63$

$P(32) \land P(63) \Rightarrow P(2016)$ (like leaping induction)

**Much Stronger Claim:**

$Q(n) : 2, 3, \ldots, n$ are all products of primes.

$P(n) : n$ is a product of primes. (Compare)

$Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n)$.

**Surprise!** The much stronger claim is much easier to prove. Also, $Q(n) \Rightarrow P(n)$.

Fundamental Theorem of Arithmetic: Proof of Part (i)

$P(n) : n$ is a product of primes.

$Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n)$.

**Proof.** (By Induction that $Q(n)$ is T for $n \geq 2$.)

1. **[Base case]** $Q(1)$ claims that 2 is a product of primes, which is clearly T.

2. **[Induction step]** Show $Q(n) \Rightarrow Q(n + 1)$ for $n \geq 2$ (direct proof).

   Assume $Q(n)$ is T: each of 2, 3, $\ldots$, $n$ are a product of primes.

   Show $Q(n + 1)$ is T: each of 2, 3, $\ldots$, $n$, $n + 1$ is a product of primes.

   Since we assumed $Q(n)$, we already have that 2, 3, $\ldots$, $n$ are products of primes.

   **To prove** $Q(n + 1)$, **we only need to prove** $n + 1$ is a product of primes.

   a. $n + 1$ is prime. Done (nothing to prove).

   b. $n + 1$ is not prime, $n + 1 = k\ell$, where $2 \leq k, \ell \leq n$.

      $P(k) \Rightarrow k$ is a product of primes.

      $P(\ell) \Rightarrow \ell$ is a product of primes.

      $n + 1 = k\ell$ is a product of primes and $Q(n + 1)$ is T.

3. By induction, $Q(n)$ is T $\forall n \geq 2$.

Every $n \geq 1$ Has a Binary Expansion

$P(n) : \text{Every } n \geq 1 \text{ is a sum of distinct powers of two (its binary expansion).}$

\[2^2 = 2^1 + 2^1 + 2^0.\]

**Base Case:** $P(1)$ is T: $1 = 2^0$

**Strong Induction:** Assume $P(1) \land P(2) \land \cdots \land P(n)$ and prove $P(n + 1)$.

If $n$ is even, then $n + 1 = 2^e + \text{binary expansion of } n$,

\[e.g. \ 23 = 2^3 + 2^2 + 2^1 + 2^0\]

If $n$ is odd, then multiply each term in the expansion of $P(n + 1)$ by 2 to get $n + 1$.

\[e.g. \ 24 = 2 \times (2^3 + 2^1) = 2^4 + 2^2\]

**Exercise.** Give the formal proof by strong induction.
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, games of strategy, . . . .

Equal Pile Nim (old English/German: to steal or pilfer)

Player 1

Player 2

P(n) : Player 2 can win the game that starts with n pennies in each row.

Equalization strategy:

Player 1

Player 2

Player 2 can always return the game to smaller equal piles.

If Player 2 wins the smaller game, Player 2 wins the larger game. That’s strong induction!

Exercise. Give the full formal proof by strong induction.

Challenge. What about more than 2 piles? What about unequal piles. (Problem 6.20).

Please, Please, Please! Become Good at Induction!

Checklist When Approaching an Induction Problem.

- Are you trying to prove a “For all . . .” claim?
- Identify the claim P(n), especially the parameter n. Here is an example.
  
  Prove: geometric mean \( \leq \) arithmetic mean. What is P(n)? What is n?
  
  P(n) : geometric mean \( \leq \) arithmetic mean for every set of n positive numbers.

Identifying the right claim is important. You may fail because you try to prove too much. Your P(n + 1) is too heavy a burden. You may fail because you try to prove too little. Your P(n) is too weak a support. You must balance the strength of your claim so that the support is just enough for the burden. — G. Polya (paraphrased).

- Tinker. Does the claim hold for small n (n = 1, 2, 3, . . .)? These become base cases.

- Tinker. Can you see why (say) P(5) follows from P(1), P(2), P(3), P(4)? This is the crux of induction; to build up from smaller n to a larger n.

- Determine the type of induction: try strong induction first.
- Write out the skeleton of the proof to see exactly what you need to prove.
- Determine and prove the base cases.
- Prove P(n + 1) in the induction step. You must use the induction hypothesis.