Foundations of Computer Science
Lecture 6

Strong Induction

Strengthening the Induction Hypothesis
Strong Induction
Many Flavors of Induction
Proving “for all”:

- $P(n) : 4^n - 1$ is divisible by 3. $\forall n : P(n)$?

- $P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$. $\forall n : P(n)$?

- $P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n + 1)(2n + 1)$. $\forall n : P(n)$?

Induction.

Induction and Well-Ordering.
Today: Twists on Induction

1. **Solving Harder Problems with Induction**
   - \( \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n} \)

2. **Strengthening the Induction Hypothesis**
   - \( n^2 < 2^n \)
   - \( L \)-tiling.

3. **Many Flavors of Induction**
   - Leaping Induction
     - Postage; \( n^3 < 2^n \)
   - Strong Induction
     - Fundamental Theorem of Arithmetic
     - Games of Strategy
A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2n$

Proof. $P(n): \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

1: [Base case] $P(1)$ claims that $1 \leq 2 \times \sqrt{1}$, which is clearly T.

2: [Induction step] Show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$ (direct proof)

Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$.

Show $P(n + 1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$.

\[
\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}}
\]

\[
\leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}}
\]

(lemma)

\[
\leq 2\sqrt{n+1}
\]

So, $P(n + 1)$ is T.

3: By induction, $P(n)$ is T $\forall n \geq 1$. 

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**Lemma.** $2\sqrt{n+1}/\sqrt{n+1} \leq 2\sqrt{n+1}$

**Proof.** By contradiction.

\[
2\sqrt{n+1}/\sqrt{n+1} > 2\sqrt{n+1}
\]

\[
\rightarrow 2\sqrt{n+1} + 1 > 2(n+1)
\]

\[
\rightarrow 4n(n+1) > (2n+1)^2
\]

\[
\rightarrow 0 > 1 \quad \text{FISHY!}
\]
Proving Stronger Claims

\[ n^2 \leq 2^n \quad \text{for } n \geq 4. \]

**Induction Step.** Must use \( n^2 \leq 2^n \) to show \((n + 1)^2 \leq 2^{n+1}\).

\[
(n + 1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \leq 2^n + 2^n = 2^{n+1}
\]

What to do with the \(2n + 1\)?

Would be fine if \(2n + 1 \leq 2^n\).

With induction, it can be easier to prove a stronger claim.
Strengthen the Claim: \( Q(n) \) Implies \( P(n) \)

\[ Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n. \]

\[ Q(4) \rightarrow Q(5) \rightarrow Q(6) \rightarrow Q(7) \rightarrow Q(8) \rightarrow Q(9) \rightarrow \cdots \]

Proof. \( Q(n) : (i) \ n^2 \leq 2^n \quad \text{AND} \quad (ii) \ 2n + 1 \leq 2^n. \)

1. [Base case] \( Q(4) \) claims \( (i) \ 4^2 \leq 2^4 \ AND \ (ii) \ 2 \times 4 + 1 \leq 2^4. \) Both clearly T.

2. [Induction step] Show \( Q(n) \rightarrow Q(n+1) \) for \( n \geq 4 \) (direct proof).

Assume (induction hypothesis) \( Q(n) \) is T: \( (i) \ n^2 \leq 2^n \ AND \ (ii) \ 2n + 1 \leq 2^n. \)

Show \( Q(n+1) \) is T: \( (i) \ (n+1)^2 \leq 2^{n+1} \ AND \ (ii) \ 2(n+1) + 1 \leq 2^{n+1}. \)

\[ (i) \quad (n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \checkmark \]

(because from the induction hypothesis \( n^2 \leq 2^n \ and \ 2n + 1 \leq 2^n \))

\[ (ii) \ 2(n+1) + 1 = 2 + 2n + 1 \leq 2^n + 2^n = 2^{n+1} \checkmark \]

(because \( 2 \leq 2^n \) and from the induction hypothesis \( 2n + 1 \leq 2^n \))

So, \( Q(n+1) \) is T.

3. By induction, \( Q(n) \) is T \( \forall n \geq 4. \)
Can you tile a $2^n \times 2^n$ patio missing a center square. You have only $\square$ – tiles?

**TINKER!**

\[ P(n) : \text{The } 2^n \times 2^n \text{ grid minus a center-square can be } L\text{-tiled}. \]
Suppose \( P(n) \) is \( \text{T} \). What about \( P(n+1) \)?

The \( 2^{n+1} \times 2^{n+1} \) patio can be decomposed into four \( 2^n \times 2^n \) patios.

**Problem.** Corner squares are missing. \( P(n) \) can be used only if center-square is missing.

**Solution.** Strengthen claim to also include patios missing corner-squares.

\[
Q(n) : \begin{align*}
(i) & \text{ The } 2^n \times 2^n \text{ grid missing a } \text{center-square} \text{ can be } L\text{-tiled}; \text{ AND} \\
(ii) & \text{ The } 2^n \times 2^n \text{ grid missing a } \text{corner-square} \text{ can be } L\text{-tiled.}
\end{align*}
\]
Assume $Q(n)$: (i) The $2^n \times 2^n$ grid missing a center-square can be $L$-tiled; AND (ii) The $2^n \times 2^n$ grid missing a corner-square can be $L$-tiled.

Induction step: Must prove two things for $Q(n + 1)$, namely (i) and (ii).

(i) Center square missing.

(ii) Corner square missing.

Your task: Add base cases and complete the formal proof.

Exercise 6.4. What if the missing square is some random square? Strengthen further.
A Tricky Induction Problem

\[ P(n) : n^3 < 2^n, \quad \text{for } n \geq 10. \]  
(Exercise 6.2)

Suppose \( P(n) \) is true. Consider \( P(n + 2) : (n + 2)^3 < 2^{n+2} \)?

\[
(n + 2)^3 = n^3 + 6n^2 + 12n + 8 \\
< n^3 + n \cdot n^2 + n^2 \cdot n + n^3 \\
= 4n^3 < 4 \cdot 2^n = 2^{n+2} \quad \text{(}n \geq 10 \rightarrow 6 < n; \ 12 < n^2; \ 8 < n^3) \]

\[ P(n) \rightarrow P(n + 2). \]

Base cases. \( P(10) : 10^3 < 2^{10} \checkmark \) \quad and \quad \( P(11) : 11^3 < 2^{11} \checkmark \)

**Leaping Induction**

**Induction.** One base case.

\[
P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots
\]

**Leaping Induction.** More than one base case.

Example. Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

<table>
<thead>
<tr>
<th>3¢</th>
<th>4¢</th>
<th>5¢</th>
<th>6¢</th>
<th>7¢</th>
<th>8¢</th>
<th>9¢</th>
<th>10¢</th>
<th>11¢</th>
<th>12¢</th>
<th>⋯</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>−</td>
<td>3,3</td>
<td>3,4</td>
<td>4,4</td>
<td>3,3,3</td>
<td>3,3,4</td>
<td>3,4,4</td>
<td>4,4,4</td>
<td>⋯</td>
</tr>
</tbody>
</table>

\[P(n) : \text{Postage of } n \text{ cents can be made using only 3¢ and 4¢ stamps.}\]

\[P(n) \rightarrow P(n + 3)\]  
(add a 3¢ stamp to \(n\))

**Base cases:** 6¢, 7¢, 8¢.

**Practice.** Exercise 6.6
Fundamental Theorem of Arithmetic

2015 = 5 \times 13 \times 31.

Theorem. (The Primes \( \mathcal{P} = \{2, 3, 5, 7, 11, \ldots \} \) are the atoms for numbers.)

Suppose \( n \geq 2 \). Then,

- \( n \) can be written as a product of factors all of which are prime.
- The representation of \( n \) as a product of primes is unique (up to reordering).

\[ P(n) : n \text{ is a product of primes.} \]

What’s the first thing we do? **TINKER!**

2016 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 7.

Wow! No similarity between the factors of 2015 and those of 2016.

**How will \( P(n) \) help us to prove \( P(n + 1) \)?**
Much “Stronger” Induction Claim

Do smaller values of $n$ help with 2016? Yes!

$$2016 = 32 \times 63$$

$$P(32) \land P(63) \rightarrow P(2016)$$

(like leaping induction)

**Much Stronger Claim:**

$$Q(n) : 2, 3, \ldots, n \text{ are all products of primes.}$$

$$P(n) : n \text{ is a product of primes.}$$

(Compare)

$$Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n).$$

**Surprise!** The much stronger claim is *much* easier to prove. Also, $Q(n) \rightarrow P(n)$. 
Fundamental Theorem of Arithmetic: Proof of Part (i)

$P(n): n$ is a product of primes.

\[ Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n). \]

**Proof.** (By Induction that $Q(n)$ is $\mathsf{T}$ for $n \geq 2$.)

1. **[Base case]** $Q(1)$ claims that 2 is a product of primes, which is clearly $\mathsf{T}$.

2. **[Induction step]** Show $Q(n) \rightarrow Q(n + 1)$ for $n \geq 2$ (direct proof).
   
   Assume $Q(n)$ is $\mathsf{T}$: each of 2, 3, \ldots, $n$ are a product of primes.
   
   Show $Q(n + 1)$ is $\mathsf{T}$: each of 2, 3, \ldots, $n$, $n+1$ is a product of primes.
   
   Since we assumed $Q(n)$, we already have that 2, 3, \ldots, $n$ are products of primes. **To prove $Q(n+1)$, we only need to prove $n+1$ is a product of primes.**
   
   - $n + 1$ is prime. Done (nothing to prove).
   - $n + 1$ is not prime, $n + 1 = k\ell$, where $2 \leq k, \ell \leq n$.
     
     $P(k) \rightarrow k$ is a product of primes.
     
     $P(\ell) \rightarrow \ell$ is a product of primes.
     
     $n + 1 = k\ell$ is a product of primes and $Q(n + 1)$ is $\mathsf{T}$.

3. By induction, $Q(n)$ is $\mathsf{T}$ $\forall n \geq 2$. 

[End of proof]
**Strong Induction.** To prove $P(n) \forall n \geq 1$ by strong induction, you use induction to prove the *stronger* claim:

$$Q(n) : \text{each of } P(1), P(2), \ldots, P(n) \text{ are T.}$$

<table>
<thead>
<tr>
<th></th>
<th><strong>Ordinary Induction</strong></th>
<th><strong>Strong Induction</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Base Case</strong></td>
<td>Prove $P(1)$</td>
<td>Prove $Q(1) = P(1)$</td>
</tr>
<tr>
<td><strong>Induction Step</strong></td>
<td>Assume: $P(n)$</td>
<td>Assume: $Q(n) = P(1) \land P(2) \land \cdots \land P(n)$</td>
</tr>
<tr>
<td></td>
<td>Prove: $P(n + 1)$</td>
<td>Prove: $P(n + 1)$</td>
</tr>
</tbody>
</table>

Strong induction is always easier.
Every \( n \geq 1 \) Has a Binary Expansion

\[ P(n) : \text{Every } n \geq 1 \text{ is a sum of distinct powers of two (its binary expansion).} \]

\[ 22 = 2^1 + 2^2 + 2^4. \]

(\( 22_{\text{binary}} = \overline{10110}_2 \))

**Base Case:** \( P(1) \) is \( 1 = 2^0 \)

**Strong Induction:** Assume \( P(1) \land P(2) \land \cdots \land P(n) \) and prove \( P(n+1) \).

If \( n \) is even, then \( n + 1 = 2^0 + \text{binary expansion of } n \),

\[ \text{e.g. } 23 = 2^0 + 2^1 + 2^2 + 2^4 \]

If \( n \) is odd, then multiply each term in the expansion of \( \frac{1}{2}(n+1) \) by 2 to get \( n + 1 \).

\[ \text{e.g. } 24 = 2 \times \overline{12} = 2^3 + 2^4 \]

**Exercise.** Give the formal proof by strong induction.
The Many Applications of Induction

Tournament rankings, greedy or recursive algorithms, games of strategy, \textit{\textbf{\ldots}}

\textbf{Equal Pile Nim} (old English/German: to steal or pilfer)

\begin{itemize}
\item Player 1
\item Player 2
\end{itemize}

\[ P(n) : \text{Player 2 can win the game that starts with } n \text{ pennies in each row.} \]

\textbf{Equalization strategy:}

\begin{itemize}
\item Player 1
\item Player 2
\end{itemize}

Player 2 can always return the game to \textit{smaller} equal piles.

If Player 2 wins the smaller game, Player 2 wins the larger game. That’s strong induction!

\textbf{Exercise.} Give the full formal proof by strong induction.

\textbf{Challenge.} What about more than 2 piles. What about unequal piles. (Problem 6.20).
Please, Please, Please! Become Good at Induction!

Checklist When Approaching an Induction Problem.

- Are you trying to prove a “For all . . .” claim?

- Identify the claim $P(n)$, especially the parameter $n$. Here is an example.
  
  **Prove:** geometric mean $\leq$ arithmetic mean. What is $P(n)$? What is $n$?
  
  $P(n)$: geometric mean $\leq$ arithmetic mean for every set of $n$ positive numbers.

  **Identifying the right claim is important.**
  You may fail because you try to prove too much. Your $P(n + 1)$ is too heavy a burden. You may fail because you try to prove too little. Your $P(n)$ is too weak a support. You must balance the strength of your claim so that the support is just enough for the burden. — G. Polya (paraphrased).

- Tinker. Does the claim hold for small $n$ ($n = 1, 2, 3, \ldots$)? These become base cases.

- Tinker. Can you see why (say) $P(5)$ follows from $P(1), P(2), P(3), P(4)$?
  This is the crux of induction; to build up from smaller $n$ to a larger $n$.

- Determine the type of induction: try strong induction first.

- Write out the skeleton of the proof to see exactly what you need to prove.

- Determine and prove the base cases.

- Prove $P(n + 1)$ in the induction step. You **must** use the induction hypothesis.