Foundations of Computer Science
Lecture 17

Independent Events

Independence is a Powerful Assumption
The Fermi Method
Coincidence and the Birthday Paradox
Application to Hashing
Random Walks and Gambler’s Ruin
Last Time

1. New information changes a probability.

2. Conditional probability.

3. Conditional probability traps.
   - Medical testing.

4. Law of total probability.
   - Case by case probability analysis.
Today: Independent Events

1. Independence is an assumption
   - Fermi method
   - Multiway independence

2. Coincidence and the birthday paradox
   - Application to hashing

3. Random walk and gambler’s ruin
Indepedence is a Simplifying Assumption

- Sex of first child has nothing to do with sex of second → independent.
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Toss two coins.

\[ P[Coin\ 1=H] = \frac{1}{2} \quad P[Coin\ 2=H] = \frac{1}{2} \quad P[Coin\ 1=H\ \text{AND}\ Coin\ 2=H] = \frac{1}{4} \]

Toss 100 times: Coin 1 \(\approx\) 50H (of these) → Coin 2 \(\approx\) 25H (independent)

\[ P[Coin\ 1=H\ \text{AND}\ Coin\ 2=H] = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P[Coin\ 1=H] \times P[Coin\ 2=H]. \]
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Toss two coins.

\[ P[\text{Coin 1}=H] = \frac{1}{2} \quad P[\text{Coin 2}=H] = \frac{1}{2} \quad P[\text{Coin 1}=H \text{ AND Coin 2}=H] = \frac{1}{4} \]

Toss 100 times: Coin 1 \( \approx 50H \) (of these) → Coin 2 \( \approx 25H \) (independent)

\[ P[\text{Coin 1}=H \text{ AND Coin 2}=H] = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P[\text{Coin 1}=H] \times P[\text{Coin 2}=H]. \]

\[ P[\text{rain AND clouds}] = P[\text{rain}] = \frac{1}{7} \gg \frac{1}{35} = P[\text{rain}] \times P[\text{clouds}]. \] (not independent)
Definition of Independence

Events $A$ and $B$ are independent if “They have nothing to do with each other.” Knowing the outcome is in $B$ does not change the probability that the outcome is in $A$. 
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The events $A$ and $B$ are independent if

$$\mathbb{P}[A \text{ AND } B] = \mathbb{P}[A \cap B] = \mathbb{P}[A] \times \mathbb{P}[B].$$

In general, $\mathbb{P}[A \cap B] = \mathbb{P}[A \mid B] \times \mathbb{P}[B]$. Independence means that

$$\mathbb{P}[A \mid B] = \mathbb{P}[A].$$
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\]
In general, $P[A \cap B] = P[A \mid B] \times P[B]$. Independence means that
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P[A \mid B] = P[A].
\]

Independence is a non-trivial assumption, and you can’t always assume it.

When you can assume independence

PROBABILITIES MULTIPLY
Fermi-Method: How Many Dateable Girls Are Out There?
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\[ A_1 = \text{“Lives nearby”;} \quad A_2 = \text{“Right sex”;} \quad A_3 = \text{“Right age”;} \quad A_4 = \text{“Single”;} \]
\[ A_5 = \text{“Educated”;} \quad A_6 = \text{“Attractive”;} \quad A_7 = \text{“Finds me attractive”;} \quad A_8 = \text{“We get along”}. \]
Fermi-Method: How Many Dateable Girls Are Out There?

\[ A = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7 \cap A_8 \]  
(all criteria must be met)
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- \( \mathbb{P}[\text{“Attractive”}] \) \quad \frac{1}{5} \text{ (you find 1 in 5 people attractive)} \n
- \( \mathbb{P}[\text{“Finds me attractive”}] \) \quad \frac{1}{10} \text{ (you are modest)}
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\mathbb{P}[\text{“Finds me attractive”}] & \quad \frac{1}{10} \quad (\text{you are modest}) \\
\mathbb{P}[\text{“We get along”}] & \quad \frac{1}{16} \quad (\text{you get along with 1 in 4 people and assume so for her})
\end{align*}
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\mathbb{P}[\text{"We get along"}] &= \frac{1}{16} \quad \text{(you get along with 1 in 4 people and assume so for her)}
\end{align*} \]

\[ \mathbb{P}[\text{"Dateable"}] = \frac{3}{1000} \times \frac{1}{2} \times \frac{15}{100} \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{5} \times \frac{1}{10} \times \frac{1}{16} \times \approx 3.5 \times 10^{-8}, \]

1-in-30 million (or 250) dateable girls.


<table>
<thead>
<tr>
<th>$\Omega$</th>
<th>HHH</th>
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$A_1 = \{\text{coins 1,2 match}\}$

$A_2 = \{\text{coins 2,3 match}\}$

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## Multiway Independence

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- $A_1 = \{$coins 1,2 match$\}$
- $A_2 = \{$coins 2,3 match$\}$
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\[ P[A_1] = P[A_2] = P[A_3] = \frac{1}{2}. \]
Multiway Independence

\[ \Omega \begin{array}{cccccccc}
\text{HHH} & \text{HHT} & \text{HTH} & \text{HTT} & \text{THH} & \text{THT} & \text{TTH} & \text{TTT} \\
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8}
\end{array} \]

\[ P(\omega) \begin{array}{cccccccc}
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8}
\end{array} \]

\[ A_1 = \{ \text{coins 1,2 match} \} \]

\[ A_2 = \{ \text{coins 2,3 match} \} \]

\[ A_3 = \{ \text{coins 1,3 match} \} \]

- \[ P[A_1] = P[A_2] = P[A_3] = \frac{1}{2} \] (independent)

- \[ P[A_1 \cap A_2] = P[A_2 \cap A_3] = P[A_1 \cap A_3] = \frac{1}{4} \] (independent)
Multiway Independence

\[ \Omega = \{ \text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} \} \]

\[
P(\omega) = \begin{bmatrix}
\frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8}
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- \( P[A_1] = P[A_2] = P[A_3] = \frac{1}{2} \).
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- \( P[A_1 \cap A_2 \cap A_3] = \frac{1}{4} \).

(1,2) match AND (2,3) match \( \rightarrow \) (1,3) match.

2-way independent, not 3-way independent.
Multiway Independence

\[ \Omega \equiv \{ \text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} \} \]

\[ P(\omega) = \{ \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8} \} \]

- \( \mathbb{P}[A_1] = \mathbb{P}[A_2] = \mathbb{P}[A_3] = \frac{1}{2} \).
- \( \mathbb{P}[A_1 \cap A_2] = \mathbb{P}[A_2 \cap A_3] = \mathbb{P}[A_1 \cap A_3] = \frac{1}{4} \).
- \( \mathbb{P}[A_1 \cap A_2 \cap A_3] = \frac{1}{4} \).

(1,2) match AND (2,3) match \( \rightarrow \) (1,3) match.

2-way independent, not 3-way independent.

- \( A_1 = \{ \text{coins 1,2 match} \} \)
- \( A_2 = \{ \text{coins 2,3 match} \} \)
- \( A_3 = \{ \text{coins 1,3 match} \} \)

\( \mathbb{P}[A_i] = \frac{1}{2} \) for \( i = 1, 2, 3 \).

\[ \mathbb{P}[A_1 \cap A_2] = \frac{1}{4} \] and \( \mathbb{P}[A_2 \cap A_3] = \frac{1}{4} \) and \( \mathbb{P}[A_1 \cap A_3] = \frac{1}{4} \).

\( \mathbb{P}[A_1 \cap A_2 \cap A_3] = \frac{1}{4} \) follows.

\( A_1, \ldots, A_n \) are independent if the probability of any intersection of distinct events is the product of the event-probabilities of those events,

\[ \mathbb{P}[A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}] = \mathbb{P}[A_{i_1}] \cdot \mathbb{P}[A_{i_2}] \cdots \mathbb{P}[A_{i_k}] . \]
Two hundred students $S = \{s_1, \ldots, s_{200}\}$,

- Birthdays are *independent* (no twins, triplets, ...) and all birthdays are equally likely.

\[ B = 366 \]
Coincidence: Let’s Try to Find a FOCS-Twin

Two hundred students $S = \{s_1, \ldots, s_{200}\}$,

- Birthdays are independent (no twins, triplets, ...) and all birthdays are equally likely.

\[
\begin{array}{cccccccc}
S_1 & & & & & & & \\
1 & 2 & 3 & 4 & \cdots & & & \\
\end{array}
\]

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\[
P[s_1 \text{ has no FOCS-twin}] = \left(\frac{B-1}{B}\right)^{N-1} = \left(\frac{365}{366}\right)^{199}
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\[
\begin{array}{cccccc}
S_1 & S_2 & & & & \\
1 & 2 & 3 & 4 & \cdots & B = 366
\end{array}
\]

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P[s_1 \text{ has no FOCS-twin}] = \left(\frac{B-1}{B}\right)^{N-1} = \left(\frac{365}{366}\right)^{199}
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\[
P[s_2 \text{ has no FOCS-twin} \mid s_1 \text{ has no FOCS-twin}] = \left(\frac{B-2}{B-1}\right)^{N-2} = \left(\frac{364}{365}\right)^{198}
\]
Coincidence: Let’s Try to Find a FOCS-Twin

Two hundred students \( S = \{s_1, \ldots, s_{200}\} \),

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\[
\begin{array}{ccccccccccccccccccc}
S_1 & S_2 & S_3 & | & | & | & | & | & | & | & | & | & & & & & & & \quad B = 366 \\
\end{array}
\]

\[
\begin{align*}
\mathbb{P} [s_1 \text{ has no FOCS-twin}] &= \left( \frac{B-1}{B} \right)^{N-1} = \left( \frac{365}{366} \right)^{199} \\
\mathbb{P} [s_2 \text{ has no FOCS-twin} \mid s_1 \text{ has no FOCS-twin}] &= \left( \frac{B-2}{B-1} \right)^{N-2} = \left( \frac{364}{365} \right)^{198}
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$$B = 366$$

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$$\Pr[s_2 \text{ has no FOCS-twin} \mid s_1 \text{ has no FOCS-twin}] = \left(\frac{B-2}{B-1}\right)^{N-2} = \left(\frac{364}{365}\right)^{198}$$

$$\Pr[s_3 \text{ has no FOCS-twin} \mid s_1, s_2 \text{ have no FOCS-twin}] = \left(\frac{B-3}{B-2}\right)^{N-3} = \left(\frac{363}{364}\right)^{197}$$
Coincidence: Let’s Try to Find a FOCS-Twin

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\]

\[
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\]

\[
P[s_3 \text{ has no FOCS-twin} \mid s_1, s_2 \text{ have no FOCS-twin}] = \left( \frac{B-3}{B-2} \right)^{N-3} = \left( \frac{363}{364} \right)^{197}
\]

\[
\vdots
\]

\[
P[s_k \text{ has no FOCS-twin} \mid s_1, \ldots, s_{k-1} \text{ have no FOCS-twin}] = \left( \frac{B-k}{B-k+1} \right)^{N-k} = \left( \frac{366-k}{366-k+1} \right)^{N-k}
\]
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\[\vdots\]

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\[
P[s_1, \ldots, s_k \text{ have no FOCS-twin}] = \left(\frac{365}{366}\right)^{199} \times \left(\frac{364}{365}\right)^{198} \times \left(\frac{363}{364}\right)^{197} \times \cdots \times \left(\frac{366-k}{366-k+1}\right)^{N-k} \approx 0.58
\]
Coincidence: Let’s Try to Find a FOCS-Twin

Two hundred students $S = \{s_1, \ldots, s_{200}\}$,

- Birthdays are independent (no twins, triplets, …) and all birthdays are equally likely.

$$B = 366$$

\[
\begin{align*}
\mathbb{P}[s_1 \text{ has no FOCS-twin}] &= \left(\frac{B-1}{B}\right)^{N-1} = \left(\frac{365}{366}\right)^{199} \\
\mathbb{P}[s_2 \text{ has no FOCS-twin} \mid s_1 \text{ has no FOCS-twin}] &= \left(\frac{B-2}{B-1}\right)^{N-2} = \left(\frac{364}{365}\right)^{198} \\
\mathbb{P}[s_3 \text{ has no FOCS-twin} \mid s_1, s_2 \text{ have no FOCS-twin}] &= \left(\frac{B-3}{B-2}\right)^{N-3} = \left(\frac{363}{364}\right)^{197} \\
&\vdots \\
\mathbb{P}[s_k \text{ has no FOCS-twin} \mid s_1, \ldots, s_{k-1} \text{ have no FOCS-twin}] &= \left(\frac{B-k}{B-k+1}\right)^{N-k} = \left(\frac{366-k}{366-k+1}\right)^{N-k} \\
\mathbb{P}[s_1, \ldots, s_k \text{ have no FOCS-twin}] &= \left(\frac{365}{366}\right)^{199} \times \left(\frac{364}{365}\right)^{198} \times \left(\frac{363}{364}\right)^{197} \times \cdots \times \left(\frac{366-k}{366-k+1}\right)^{N-k} \approx 0.58
\end{align*}
\]

| Finding a FOCS-twin by the $k$th student with class size 200 |
|-----------------|------|------|------|------|------|------|------|------|------|------|
| $k$             | 1    | 2    | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
| chances (%)     | 42.0 | 66.3 | 80.4 | 88.6 | 93.3 | 96.1 | 97.7 | 98.7 | 99.2 | 99.5 |
|                 | 23   | 25   |      |      |      |      |      |      |      |      |
|                 | 100  |      |      |      |      |      |      |      |      |      |
The Birthday Paradox

In a party of 50 people, what are the chances that two have the same birthday?
The Birthday Paradox

In a party of 50 people, what are the chances that two have the same birthday?

Same as asking for

\[ \Pr[s_1, \ldots, s_{50} \text{ have no FOCS-twin}]. \]
The Birthday Paradox

In a party of 50 people, what are the chances that two have the same birthday?

Same as asking for

$$P[s_1, \ldots, s_{50} \text{ have no FOCS-twin}].$$

Answer:

$$P[ \text{no social twins}] = \left(\frac{365}{366}\right)^{49} \times \left(\frac{364}{365}\right)^{48} \times \left(\frac{363}{364}\right)^{47} \times \cdots \times \left(\frac{315}{316}\right)^0 \approx 0.03.$$
The Birthday Paradox

In a party of 50 people, what are the chances that two have the same birthday?

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Chances are about 97% that two people share a birthday!

**Moral:** when searching for something among many options (1225 pairs of people), *do not be surprised* when you find it.
Example Queries

\[
\begin{align*}
\text{search(apples)} &= \{\text{page.1, page.2}\} \\
\text{search(hate)} &= \{\text{page.2, page.3}\} \\
\text{search(bananas)} &= \{\text{page.3}\}
\end{align*}
\]
Search and Hashing

Example Queries

```
search(apples) = \{page.1, page.2\}
search(hate) = \{page.2, page.3\}
search(bananas) = \{page.3\}
```

Web-address Directory

<table>
<thead>
<tr>
<th>Item</th>
<th>Directory</th>
</tr>
</thead>
<tbody>
<tr>
<td>apples</td>
<td>{page.1, page.2}</td>
</tr>
<tr>
<td>bananas</td>
<td>{page.3}</td>
</tr>
<tr>
<td>dirty</td>
<td>{page.1, page.2}</td>
</tr>
<tr>
<td>freaks</td>
<td>{page.2}</td>
</tr>
<tr>
<td>hate</td>
<td>{page.2, page.3}</td>
</tr>
<tr>
<td>health</td>
<td>{page.1, page.2}</td>
</tr>
<tr>
<td>hurt</td>
<td>{page.1}</td>
</tr>
<tr>
<td>people</td>
<td>{page.3}</td>
</tr>
<tr>
<td>survey</td>
<td>{page.3}</td>
</tr>
</tbody>
</table>

😃 \(O(\log N)\) search 😞
Search and Hashing

Example Queries

- \text{search(apples)} = \{\text{page.1, page.2}\}
- \text{search(hate)} = \{\text{page.2, page.3}\}
- \text{search(bananas)} = \{\text{page.3}\}

Hash words into a table (array) using a hash function \( H(w) \), e.g:

\[
H(\text{hate}) = 8^{17} + 1^{17} + 20^{17} + 5^{17} \pmod{11} = 7
\]

<table>
<thead>
<tr>
<th>Index</th>
<th>Hashed Word</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>bananas → {page.3}</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>hurt → {page.1}</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>people → {page.3}</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>dirty → {page.1, page.2}</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>freaks → {page.2}</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>hate → {page.2, page.3}</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>apples → {page.1, page.2}</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>survey → {page.3}</td>
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<td></td>
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<td></td>
</tr>
</tbody>
</table>

\( O(\log N) \) search
Search and Hashing

Example Queries

| search(apples) = \{page.1, page.2\} |
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\]

search\( (w) \): GOTO hash-table row \( H(w) \).

Web-address Directory

| apples \rightarrow \{page.1, page.2\} |
| bananas \rightarrow \{page.3\}        |
| dirty \rightarrow \{page.1, page.2\}  |
| freaks \rightarrow \{page.2\}         |
| hate \rightarrow \{page.2, page.3\}   |
| health \rightarrow \{page.1, page.2\} |
| hurt \rightarrow \{page.1\}           |
| people \rightarrow \{page.3\}         |
| survey \rightarrow \{page.3\}         |

\( O(\log N) \) search
Hash words into a table (array) using a hash function $H(w)$, e.g:

$$H(\text{hate}) = 8^{17} + 1^{17} + 20^{17} + 5^{17} \pmod{11} = 7$$

search($w$): GOTO hash-table row $H(w)$.

Collisions: (hate, freaks), (survey, apples)
Problem: What if you search for hate or survey?
Search and Hashing

Example Queries

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| search(hate) = {page.2, page.3} |
| search(bananas) = {page.3} |

Hash words into a table (array) using a hash function \( H(w) \), e.g:

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\]

search\((w)\): GOTO hash-table row \( H(w) \).

Collisions: (hate,freaks), (survey,apples)
Problem: What if you search for hate or survey?

Good hash function maps words independently and randomly. No collisions \( \rightarrow O(1) \) search (constant time, not log \( N \)).
Words $w_1, w_2, \ldots, w_N$ and Hashing $\leftrightarrow$ Students $s_1, s_2, \ldots, s_N$ and Birthdays
Words $w_1, w_2 \ldots, w_N$ and Hashing $w_1, \ldots, w_N$ HASHED to rows $0, 1, \ldots, B - 1$ \iff Students $s_1, s_2, \ldots, s_N$ and Birthdays $s_1, \ldots, s_N$ BORN on days $0, 1, \ldots, B - 1$
Hashing and FOCS-twins

Words $w_1, w_2, \ldots, w_N$ and Hashing $\leftrightarrow$ Students $s_1, s_2, \ldots, s_N$ and Birthdays

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No collisions, or HASH-twins $\leftrightarrow$ No FOCS-twins
## Hashing and FOCS-twins

### Table

<table>
<thead>
<tr>
<th>Words $w_1, w_2, \ldots, w_N$ and Hashing</th>
<th>↔</th>
<th>Students $s_1, s_2, \ldots, s_N$ and Birthdays</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1, \ldots, w_N$ HASHED to rows $0, 1, \ldots, B - 1$</td>
<td>↔</td>
<td>$s_1, \ldots, s_N$ BORN on days $0, 1, \ldots, B - 1$</td>
</tr>
<tr>
<td>No collisions, or HASH-twins</td>
<td>↔</td>
<td>No FOCS-twins</td>
</tr>
</tbody>
</table>

### Example

Suppose you have $N = 10$ words $w_1, w_2, \ldots, w_{10}$.

$B = 10$ (hash table has as many rows as words).
Hashing and FOCS-twins

Words $w_1, w_2 \ldots, w_N$ and Hashing $\leftrightarrow$ Students $s_1, s_2, \ldots, s_N$ and Birthdays

$w_1, \ldots, w_N$ hashed to rows $0, 1, \ldots, B - 1$ $\leftrightarrow$ $s_1, \ldots, s_N$ born on days $0, 1, \ldots, B - 1$

No collisions, or HASH-twins $\leftrightarrow$ No FOCS-twins

Example: Suppose you have $N = 10$ words $w_1, w_2, \ldots, w_{10}$.

$B = 10$ (hash table has as many rows as words).

$$
\Pr[\text{no collisions}] = \left(\frac{9}{10}\right)^9 \times \left(\frac{8}{9}\right)^8 \times \left(\frac{7}{8}\right)^7 \times \left(\frac{6}{7}\right)^6 \times \left(\frac{5}{6}\right)^5 \times \left(\frac{4}{5}\right)^4 \times \left(\frac{3}{4}\right)^3 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{0}{1}\right)^0 \approx 0.0004.
$$
Hashing and FOCS-twins

Words $w_1, w_2, \ldots, w_N$ and Hashing $\leftrightarrow$ Students $s_1, s_2, \ldots, s_N$ and Birthdays

$w_1, \ldots, w_N$ hashed to rows $0, 1, \ldots, B - 1$ $\leftrightarrow$ $s_1, \ldots, s_N$ born on days $0, 1, \ldots, B - 1$

No collisions, or HASH-twins $\leftrightarrow$ No FOCS-twins

Example: Suppose you have $N = 10$ words $w_1, w_2, \ldots, w_{10}$.

$B = 10$ (hash table has as many rows as words).

\[
P[\text{no collisions}] = \left(\frac{9}{10}\right)^9 \times \left(\frac{8}{9}\right)^8 \times \left(\frac{7}{8}\right)^7 \times \left(\frac{6}{7}\right)^6 \times \left(\frac{5}{6}\right)^5 \times \left(\frac{4}{5}\right)^4 \times \left(\frac{3}{4}\right)^3 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{0}{1}\right)^0 \approx 0.0004.\]

$B = 20$ (hash table has as twice many rows as words).

\[
P[\text{no collisions}] = \left(\frac{19}{20}\right)^9 \times \left(\frac{18}{19}\right)^8 \times \left(\frac{17}{18}\right)^7 \times \left(\frac{16}{17}\right)^6 \times \left(\frac{15}{16}\right)^5 \times \left(\frac{14}{15}\right)^4 \times \left(\frac{13}{14}\right)^3 \times \left(\frac{12}{13}\right)^2 \times \left(\frac{11}{12}\right)^1 \times \left(\frac{10}{11}\right)^0 \approx 0.07.\]
Hashing and FOCS-twins

Words $w_1, w_2 \ldots, w_N$ and Hashing $\leftrightarrow$ Students $s_1, s_2, \ldots, s_N$ and Birthdays

$w_1, \ldots, w_N$ hashed to rows 0, 1, \ldots, $B - 1$ $\leftrightarrow$ $s_1, \ldots, s_N$ born on days 0, 1, \ldots, $B - 1$

No collisions, or HASH-twins $\leftrightarrow$ No FOCS-twins

Example: Suppose you have $N = 10$ words $w_1, w_2, \ldots, w_{10}$.

$B = 10$ (hash table has as many rows as words).

\[ \Pr[\text{no collisions}] = \left( \frac{9}{10} \right)^9 \times \left( \frac{8}{9} \right)^8 \times \left( \frac{7}{8} \right)^7 \times \left( \frac{6}{7} \right)^6 \times \left( \frac{5}{6} \right)^5 \times \left( \frac{4}{5} \right)^4 \times \left( \frac{3}{4} \right)^3 \times \left( \frac{2}{3} \right)^2 \times \left( \frac{1}{2} \right)^1 \times \left( \frac{0}{1} \right)^0 \approx 0.0004. \]

$B = 20$ (hash table has as twice many rows as words).

\[ \Pr[\text{no collisions}] = \left( \frac{19}{20} \right)^9 \times \left( \frac{18}{19} \right)^8 \times \left( \frac{17}{18} \right)^7 \times \left( \frac{16}{17} \right)^6 \times \left( \frac{15}{16} \right)^5 \times \left( \frac{14}{15} \right)^4 \times \left( \frac{13}{14} \right)^3 \times \left( \frac{12}{13} \right)^2 \times \left( \frac{11}{12} \right)^1 \times \left( \frac{10}{11} \right)^0 \approx 0.07. \]

<table>
<thead>
<tr>
<th>$B$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pr[\text{no collisions}]$</td>
<td>0.0004</td>
<td>0.07</td>
<td>0.18</td>
<td>0.29</td>
<td>0.38</td>
<td>0.45</td>
<td>0.51</td>
<td>0.56</td>
<td>0.60</td>
<td>0.63</td>
<td>0.91</td>
<td>0.96</td>
</tr>
</tbody>
</table>

$B$ large enough $\rightarrow$ chances of no collisions are high (that’s good). How large should $B$ be?
Hashing and FOCS-twins

Words $w_1, w_2 \ldots, w_N$ and Hashing $\leftrightarrow$ Students $s_1, s_2, \ldots, s_N$ and Birthdays

$w_1, \ldots, w_N$ HASHED to rows $0, 1, \ldots, B - 1$ $\leftrightarrow$ $s_1, \ldots, s_N$ BORN on days $0, 1, \ldots, B - 1$

No collisions, or HASH-twins $\leftrightarrow$ No FOCS-twins

Example: Suppose you have $N = 10$ words $w_1, w_2, \ldots, w_{10}$.

$B = 10$ (hash table has as many rows as words).

$$\Pr[\text{no collisions}] = \left(\frac{9}{10}\right)^9 \times \left(\frac{8}{9}\right)^8 \times \left(\frac{7}{8}\right)^7 \times \left(\frac{6}{7}\right)^6 \times \left(\frac{5}{6}\right)^5 \times \left(\frac{4}{5}\right)^4 \times \left(\frac{3}{4}\right)^3 \times \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{0}{1}\right)^0 \approx 0.0004.$$

$B = 20$ (hash table has as twice many rows as words).

$$\Pr[\text{no collisions}] = \left(\frac{19}{20}\right)^9 \times \left(\frac{18}{19}\right)^8 \times \left(\frac{17}{18}\right)^7 \times \left(\frac{16}{17}\right)^6 \times \left(\frac{15}{16}\right)^5 \times \left(\frac{14}{15}\right)^4 \times \left(\frac{13}{14}\right)^3 \times \left(\frac{12}{13}\right)^2 \times \left(\frac{11}{12}\right)^1 \times \left(\frac{10}{11}\right)^0 \approx 0.07.$$

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<td>0.45</td>
<td>0.51</td>
<td>0.56</td>
<td>0.60</td>
<td>0.63</td>
<td>0.91</td>
<td>0.96</td>
</tr>
</tbody>
</table>

$B$ large enough $\rightarrow$ chances of no collisions are high (that’s good). How large should $B$ be?

**Theorem.** If $B \in \omega(N^2)$, then $\Pr[\text{no collisions}] \rightarrow 1$
Random Walk: What are the Chances the Drunk Gets Home?

\[ \frac{1}{2} \quad \frac{1}{2} \]

BAR

Creator: Malik Magdon-Ismail
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

\[
\begin{array}{ccccccc}
L & RLL & RLRLL & RLRLRLL & RLRLRLRLL & \cdots \\
\frac{1}{2} & \left( \frac{1}{2} \right)^{3} & \left( \frac{1}{2} \right)^{5} & \left( \frac{1}{2} \right)^{7} & \left( \frac{1}{2} \right)^{9} & \cdots \\
\end{array}
\]

\[
P((RL)^iL) = \left( \frac{1}{2} \right)^{2i+1}
\]
**Random Walk: What are the Chances the Drunk Gets Home?**

**Infinite Outcome Tree**

Sequences leading to home:

\[
\begin{array}{cccccc}
L & RLL & RLRL & RLRLRL & RLRLRLRL & \cdots \\
\frac{1}{2} & \left(\frac{1}{2}\right)^3 & \left(\frac{1}{2}\right)^5 & \left(\frac{1}{2}\right)^7 & \left(\frac{1}{2}\right)^9 & \cdots \\
\end{array}
\]

\[
P((RL)^iL) = \left(\frac{1}{2}\right)^{2i+1}
\]

\[
\mathbb{P}[\text{home}] = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^9 + \cdots
\]

\[
= \frac{\frac{1}{2}}{1-\frac{1}{4}}
\]

\[
= \frac{2}{3}.
\]
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

\[
\begin{align*}
L & \quad RLL & \quad RLRLL & \quad RLRLRLL & \quad RLRLRLRLL & \cdots \\
\frac{1}{2} & \quad (\frac{1}{2})^3 & \quad (\frac{1}{2})^5 & \quad (\frac{1}{2})^7 & \quad (\frac{1}{2})^9 & \cdots 
\end{align*}
\]

\[P((RL)^i L) = (\frac{1}{2})^{2i+1}\]

\[P[\text{home}] = \frac{1}{2} + (\frac{1}{2})^3 + (\frac{1}{2})^5 + (\frac{1}{2})^7 + (\frac{1}{2})^9 + \cdots = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}.
\]
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

- L, RLL, RLRL, RLRLRR, RLRLRRRL, ...
- \( \frac{1}{2}, (\frac{1}{2})^3, (\frac{1}{2})^5, (\frac{1}{2})^7, (\frac{1}{2})^9, \ldots \)

\[ P((RL)^iL) = (\frac{1}{2})^{2i+1} \]

\[ P[\text{home}] = \frac{1}{2} + (\frac{1}{2})^3 + (\frac{1}{2})^5 + (\frac{1}{2})^7 + (\frac{1}{2})^9 + \cdots \]

\[ = \frac{\frac{1}{2}}{1-\frac{1}{4}} \]

\[ = \frac{2}{3}. \]

Total Probability

\[ P[\text{home}] = P[L] \cdot P[\text{home} | L] \]
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

\[
\begin{align*}
L & \quad RLL & \quad RLRLL & \quad RLRLRLL & \quad RLRLRLRLL & \quad \cdots \\
\frac{1}{2} & \quad \left(\frac{1}{2}\right)^3 & \quad \left(\frac{1}{2}\right)^5 & \quad \left(\frac{1}{2}\right)^7 & \quad \left(\frac{1}{2}\right)^9 & \quad \cdots \\
\end{align*}
\]

\[P((RL)^iL) = \left(\frac{1}{2}\right)^{2i+1}\]

\[P[\text{home}] = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^9 + \cdots\]

\[= \frac{\frac{1}{2}}{1-\frac{1}{2}}\]

\[= \frac{2}{3}.
\]

Total Probability

\[P[\text{home}] = P[L] \cdot P[\text{home} | L] + P[RR] \cdot P[\text{home} | RR]\]
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

\[
\begin{align*}
&L, RLL, RLRLL, RLRLRLL, RLRLRLRLL, \ldots \\
&\frac{1}{2}, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^5, \left(\frac{1}{2}\right)^7, \left(\frac{1}{2}\right)^9, \ldots
\end{align*}
\]

\[P((RL)^iL) = \left(\frac{1}{2}\right)^{2i+1}\]

\[P[\text{home}] = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^9 + \cdots
\]

\[= \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{2}{3}.
\]

Total Probability

\[
\begin{align*}
P[\text{home}] &= P[L] \cdot P[\text{home} | L] \\
&\quad + P[RR] \cdot P[\text{home} | RR] \\
&\quad + P[RL] \cdot P[\text{home} | RL]
\end{align*}
\]
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

L  RLL  RLRLL  RLRLRLL  RLRLRLRLL ...  
\( \frac{1}{2} \)  \( \left( \frac{1}{2} \right)^3 \)  \( \left( \frac{1}{2} \right)^5 \)  \( \left( \frac{1}{2} \right)^7 \)  \( \left( \frac{1}{2} \right)^9 \)  ...  

\( P((RL)^iL) = \left( \frac{1}{2} \right)^{2i+1} \)

\[ P[\text{home}] = \frac{1}{2} + \left( \frac{1}{2} \right)^3 + \left( \frac{1}{2} \right)^5 + \left( \frac{1}{2} \right)^7 + \left( \frac{1}{2} \right)^9 + \cdots \]

\[ = \frac{\frac{1}{2}}{1 - \frac{1}{4}} \]

\[ = \frac{2}{3}. \]

Total Probability

\[ P[\text{home}] = P[L] \cdot P[\text{home} | L] \leftarrow \frac{1}{2} \times 1 \]

\[ + P[RR] \cdot P[\text{home} | RR] \]

\[ + P[RL] \cdot P[\text{home} | RL] \]
### Infinite Outcome Tree

Sequences leading to home:

\[
\begin{array}{cccccc}
L & RLL & RLRLL & RLRLRLL & RLRLRLRLL & \ldots \\
\frac{1}{2} & (\frac{1}{2})^3 & (\frac{1}{2})^5 & (\frac{1}{2})^7 & (\frac{1}{2})^9 & \ldots \\
\end{array}
\]

\[P((RL)^i L) = (\frac{1}{2})^{2i+1}\]

\[P[\text{home}] = \frac{1}{2} + (\frac{1}{2})^3 + (\frac{1}{2})^5 + (\frac{1}{2})^7 + (\frac{1}{2})^9 + \cdots\]

\[= \frac{\frac{1}{2}}{1 - \frac{1}{2}}\]

\[= \frac{2}{3}.\]

### Total Probability

\[P[\text{home}] = P[L] \cdot P[\text{home} | L] \leftarrow \frac{1}{2} \times 1\]

\[+ P[RR] \cdot P[\text{home} | RR] \leftarrow \frac{1}{4} \times 0\]

\[+ P[RL] \cdot P[\text{home} | RL]\]
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

\[
\begin{align*}
L & \quad RLL & \quad RLRLL & \quad RLRLRLL & \cdots \\
\frac{1}{2} & \quad (\frac{1}{2})^3 & \quad (\frac{1}{2})^5 & \quad (\frac{1}{2})^7 & \quad (\frac{1}{2})^9 & \cdots
\end{align*}
\]

\[
P((RL)^i L) = (\frac{1}{2})^{2i+1}
\]

\[
P[\text{home}] = \frac{1}{2} + (\frac{1}{2})^3 + (\frac{1}{2})^5 + (\frac{1}{2})^7 + (\frac{1}{2})^9 + \cdots
\]

\[
= \frac{\frac{1}{2}}{1 - \frac{1}{4}}
\]

\[
= \frac{2}{3}.
\]

Total Probability

\[
P[\text{home}] = P[L] \cdot P[\text{home} | L] \quad \leftarrow \frac{1}{2} \times 1
\]

\[
+ P[RR] \cdot P[\text{home} | RR] \quad \leftarrow \frac{1}{4} \times 0
\]

\[
+ P[RL] \cdot P[\text{home} | RL] \quad \leftarrow \frac{1}{4} \times P[\text{home}]
\]
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

\[
\begin{align*}
L & \quad RLL & \quad RLRLL & \quad RLRLRLL & \quad RLRLRLRLL & \cdots \\
\frac{1}{2} & \quad \left(\frac{1}{2}\right)^3 & \quad \left(\frac{1}{2}\right)^5 & \quad \left(\frac{1}{2}\right)^7 & \quad \left(\frac{1}{2}\right)^9 & \cdots \\
\end{align*}
\]

\[
P((RL)^iL) = \left(\frac{1}{2}\right)^{2i+1}
\]

\[
P[\text{home}] = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^9 + \cdots
\]

\[
= \frac{\frac{1}{2}}{1 - \frac{1}{2}}
\]

\[
= \frac{2}{3}
\]

Total Probability

\[
P[\text{home}] = P[L] \cdot P[\text{home} | L] \quad \leftarrow \frac{1}{2} \times 1
\]

\[
+ P[RR] \cdot P[\text{home} | RR] \quad \leftarrow \frac{1}{4} \times 0
\]

\[
+ P[RL] \cdot P[\text{home} | RL] \quad \leftarrow \frac{1}{4} \times P[\text{home}]
\]

\[
= \frac{1}{2} + \frac{1}{4} P[\text{home}].
\]
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:
\[
\begin{array}{cccccc}
L & RLL & RLRLL & RLRLRLL & RLRLRLRLL & \cdots \\
\frac{1}{2} & (\frac{1}{2})^3 & (\frac{1}{2})^5 & (\frac{1}{2})^7 & (\frac{1}{2})^9 & \cdots \\
\end{array}
\]

\[P((RL)^i L) = (\frac{1}{2})^{2i+1}\]

\[P[\text{home}] = \frac{1}{2} + (\frac{1}{2})^3 + (\frac{1}{2})^5 + (\frac{1}{2})^7 + (\frac{1}{2})^9 + \cdots \]
\[= \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}.\]

Total Probability

\[
P[\text{home}] = P[L] \cdot P[\text{home} \mid L] \quad \leftarrow \frac{1}{2} \times 1
\]
\[+ P[RR] \cdot P[\text{home} \mid RR] \quad \leftarrow \frac{1}{4} \times 0
\]
\[+ P[RL] \cdot P[\text{home} \mid RL] \quad \leftarrow \frac{1}{4} \times P[\text{home}]
\]
\[= \frac{1}{2} + \frac{1}{4} P[\text{home}].\]

That is, \((1 - \frac{1}{4}) P[\text{home}] = \frac{1}{2}\). Solve for \(P[\text{home}]\):
Random Walk: What are the Chances the Drunk Gets Home?

Infinite Outcome Tree

Sequences leading to home:

\[
\begin{align*}
&L \quad \text{RLL} \quad \text{RLRLL} \quad \text{RLRLRLL} \quad \text{RLRLRLRLL} \quad \cdots \\
&\frac{1}{2} \quad \left(\frac{1}{2}\right)^3 \quad \left(\frac{1}{2}\right)^5 \quad \left(\frac{1}{2}\right)^7 \quad \left(\frac{1}{2}\right)^9 \quad \cdots
\end{align*}
\]

\[P((\text{RL})^n \cdot \text{L}) = \left(\frac{1}{2}\right)^{2n+1}\]

\[P[\text{home}] = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^9 + \cdots
\]

\[= \frac{\frac{1}{2}}{1 - \frac{1}{4}}
\]

\[= \frac{2}{3}.
\]

Total Probability

\[P[\text{home}] = P[L] \cdot P[\text{home} \mid L] \leftarrow \frac{1}{2} \times 1
\]

\[+ P[RR] \cdot P[\text{home} \mid RR] \leftarrow \frac{1}{4} \times 0
\]

\[+ P[RL] \cdot P[\text{home} \mid RL] \leftarrow \frac{1}{4} \times P[\text{home}]
\]

\[= \frac{1}{2} + \frac{1}{4} P[\text{home}].
\]

That is, \((1 - \frac{1}{4}) P[\text{home}] = \frac{1}{2}\). Solve for \(P[\text{home}]\):

\[P[\text{home}] = \frac{\frac{1}{2}}{1 - \frac{1}{4}}
\]

\[= \frac{2}{3}.
\]
$P_i$ is the probability to win in the game if you have $i$. 

$q = 0.6$  

$p = 0.4$
Doubling Up: A Random Walk at the Casino

$P_0 = 0$

$P_4 = 1$

$q = 0.6$

$p = 0.4$

$P_i$ is the probability to win in the game if you have $i$. 
Doubling Up: A Random Walk at the Casino

$P_0 = 0 \quad P_1 =? \quad P_4 = 1$

$\begin{array}{cccc}
$0$ & $1$ & $2$ & $3$ & $4$
\end{array}$

$q = 0.6 \quad \text{START} \quad p = 0.4$

$P_i$ is the probability to win in the game if you have $\$i$.

$P_1$
$P_i$ is the probability to win in the game if you have $i$.

\[ P_1 = qP_0 + pP_2 \]

← total expectation
$P_i$ is the probability to win in the game if you have $i$.

$$P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation}$$
Doubling Up: A Random Walk at the Casino

$P_0 = 0$

$P_1 = qP_0 + pP_2 = pP_2.$ ← total expectation

$P_2$

$P_4 = 1$

$q = 0.6$

$p = 0.4$

$P_i$ is the probability to win in the game if you have $i$. 

$q$ is the probability to lose and $p$ the probability to win. 

Creator: Malik Magdon-Ismail

Independent Events: 13/13
Doubling Up: A Random Walk at the Casino

$P_0 = 0 \quad P_1 =? \quad P_2 =? \quad P_3 =? \quad P_4 = 1$

$\$0 \quad \$1 \quad \$2 \quad \$3 \quad \$4$

$q = 0.6 \quad \text{START} \quad p = 0.4$

$P_i$ is the probability to win in the game if you have $\$i$.

\[
P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation}
\]

\[
P_2 = qP_1 + pP_3
\]
$P_i$ is the probability to win in the game if you have $\$i$.

\begin{align*}
    P_1 &= qP_0 + pP_2 = pP_2, \\
    P_2 &= qP_1 + pP_3 = pqP_2 + pP_3
\end{align*}
Doubling Up: A Random Walk at the Casino

$P_0 = 0 \quad P_1 = ? \quad P_2 = ? \quad P_4 = 1$

$q = 0.6 \quad \text{START} \quad p = 0.4 \quad $1

$P_i$ is the probability to win in the game if you have $i$.

\[
P_1 = qP_0 + pP_2 = pP_2. \quad \text{← total expectation}
\]

\[
P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \rightarrow P_2 = \frac{pP_3}{1 - pq}.
\]
\( P_i \) is the probability to win in the game if you have \$i.

\[
P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation}
\]

\[
P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \rightarrow P_2 = \frac{pP_3}{1 - pq}.
\]
Doubling Up: A Random Walk at the Casino

\[ P_0 = 0 \]
\[ P_1 = \] $1$
\[ P_2 = \] $2$
\[ P_3 = \] $3$
\[ P_4 = 1 \]

\[ q = 0.6 \quad \text{START} \quad p = 0.4 \]

\( P_i \) is the probability to win in the game if you have $i$.

\[ P_1 = qP_0 + pP_2 = pP_2. \] ← total expectation

\[ P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \rightarrow P_2 = \frac{pP_3}{1 - pq}. \]

\[ P_3 = qP_2 + pP_4 \]
Doubling Up: A Random Walk at the Casino

$P_0 = 0$  $P_1 = ?$  $P_2 = ?$  $P_3 = ?$  $P_4 = 1$

$0$  $1$  $2$  $3$  $4$

$q = 0.6$  $p = 0.4$

$P_i$ is the probability to win in the game if you have $i$.

\[
P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation}
\]

\[
P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \quad \rightarrow \quad P_2 = \frac{pP_3}{1 - pq}.
\]

\[
P_3 = qP_2 + pP_4 = \frac{pqP_3}{1 - pq} + p
\]
$P_i$ is the probability to win in the game if you have $\$i$.

\[
P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation}
\]

\[
P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \rightarrow P_2 = \frac{pP_3}{1 - pq}.
\]

\[
P_3 = qP_2 + pP_4 = \frac{pqP_3}{1 - pq} + p \rightarrow P_3 = \frac{p(1 - pq)}{1 - 2pq}.
\]
Doubling Up: A Random Walk at the Casino

$P_0 = 0 \quad P_1 = ? \quad P_2 = ? \quad P_3 = ? \quad P_4 = 1$

$\begin{array}{c}
\text{START} \\
q = 0.6 \quad p = 0.4 \\
\end{array}$

$P_i$ is the probability to win in the game if you have $i$.

\[ P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation} \]

\[ P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \rightarrow P_2 = \frac{pP_3}{1 - pq}. \]

\[ P_3 = qP_2 + pP_4 = \frac{pqP_3}{1 - pq} + p \rightarrow P_3 = \frac{p(1 - pq)}{1 - 2pq}. \]

Conclusion:

\[ P_2 = \frac{p^2}{1 - 2pq} \approx 0.31 \quad (69\% \text{ chances of RUIN}) \]
$P_0 = 0 \quad P_1 =? \quad P_2 =? \quad P_3 =? \quad P_4 = 1$

$q = 0.6 \quad \text{START} \quad p = 0.4$

$P_i$ is the probability to win in the game if you have $i$.

\[
P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation}
\]

\[
P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \rightarrow P_2 = \frac{pP_3}{1 - pq}.
\]

\[
P_3 = qP_2 + pP_4 = \frac{pqP_3}{1 - pq} + p \rightarrow P_3 = \frac{p(1 - pq)}{1 - 2pq}.
\]

Conclusion:

\[
P_2 = \frac{p^2}{1 - 2pq} \approx 0.31 \quad (69\% \text{ chances of RUIN})
\]

**Exercise.**

- What if you are trying to double up from $3$? \quad (Answer: 77\% chance of RUIN).
- What if you are trying to double up from $10$? \quad (Answer: 98\% chance of RUIN).
Doubling Up: A Random Walk at the Casino

$\begin{array}{c|c|c|c|c|c}
P_0 & P_1 & P_2 & P_3 & P_4 \\
\hline
0 & ? & ? & ? & 1 \\
\hline
\$0 & \$1 & \$2 & \$3 & \$4 \\
\end{array}$

$q = 0.6 \quad p = 0.4$

$P_i$ is the probability to win in the game if you have $\$i$.

\[
P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation}
\]

\[
P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \rightarrow P_2 = \frac{pP_3}{1 - pq}.
\]

\[
P_3 = qP_2 + pP_4 = \frac{pqP_3}{1 - pq} + p \rightarrow P_3 = \frac{p(1 - pq)}{1 - 2pq}.
\]

Conclusion:

\[
P_2 = \frac{p^2}{1 - 2pq} \approx 0.31 \quad (69\% \text{ chances of RUIN})
\]

Exercise.

- What if you are trying to double up from $\$3$?
  \hspace{1cm} (Answer: 77\% chance of RUIN).
- What if you are trying to double up from $\$10$?
  \hspace{1cm} (Answer: 98\% chance of RUIN).
$P_i$ is the probability to win in the game if you have $i$.

\[ P_1 = qP_0 + pP_2 = pP_2. \quad \leftarrow \text{total expectation} \]

\[ P_2 = qP_1 + pP_3 = pqP_2 + pP_3 \rightarrow P_2 = \frac{pP_3}{1 - pq}. \]

\[ P_3 = qP_2 + pP_4 = \frac{pqP_3}{1 - pq} + p \rightarrow P_3 = \frac{p(1 - pq)}{1 - 2pq}. \]

**Conclusion:**

\[ P_2 = \frac{p^2}{1 - 2pq} \approx 0.31 \quad (69\% \text{ chances of RUIN}) \]

**Exercise.**

- What if you are trying to double up from $3$? (Answer: 77% chance of RUIN).
- What if you are trying to double up from $10$? (Answer: 98% chance of RUIN).

The *richer* the Gambler, the greater the chances of RUIN!