Foundations of Computer Science
Lecture 20

Expected Value of a Sum

Linearity of Expectation
Iterated Expectation
Build-Up Expectation
Sum of Indicators
Last Time

1. Sample average and expected value.

2. Definition of Mathematical expectation.

3. Examples: Sum of dice; Bernoulli; Uniform; Binomial; waiting time;


5. Law of Total Expectation.
Today: Expected Value of a Sum

1. Expected value of a sum.
   - Sum of dice.
   - Binomial.
   - Waiting time.
   - Coupon collecting.

2. Iterated expectation.

3. Build-up expectation.

4. Expected value of a product.

5. Sum of indicators.
You expect to win twice as much from two lottery tickets as from one.
Expected Value of a Sum

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The expected value of a sum is a sum of the expected values.
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**The expected value of a sum is a sum of the expected values.**

**Theorem (Linearity of Expectation).** Let $X_1, X_2, \ldots, X_k$ be random variables and let $Z = a_1 X_1 + a_2 X_2 + \cdots + a_k X_k$ be a linear combination of the $X_i$. Then,
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\mathbb{E}[Z] = \mathbb{E}[a_1X_1 + a_2X_2 + \cdots + a_kX_k] = a_1 \mathbb{E}[X_1] + a_2 \mathbb{E}[X_2] + \cdots + a_k \mathbb{E}[X_k].
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**Theorem (Linearity of Expectation).** Let $X_1, X_2, \ldots, X_k$ be random variables and let $Z = a_1X_1 + a_2X_2 + \cdots + a_kX_k$ be a *linear* combination of the $X_i$. Then,

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$$

**Proof.**

\[
\mathbb{E}[Z] = \sum_{\omega \in \Omega} \left( a_1X_1(\omega) + a_2X_2(\omega) + \cdots + a_kX_k(\omega) \right) \cdot P(\omega)
\]
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$$= a_1 E[X_1] + a_2 E[X_2] + \cdots + a_k E[X_k].$$

1. Summation can be taken inside or pulled outside an expectation.
2. Constants can be taken inside or pulled outside an expectation.

$$E \left[ \sum_{i=1}^{k} a_iX_i \right] = \sum_{i=1}^{k} a_i E[X_i]$$
Let $X$ be the sum of 4 fair dice, what is $\mathbb{E}[X]$?
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| sum | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\mathbb{P}[\text{sum}]$ | $\frac{1}{1296}$ | $\frac{4}{1296}$ | $\frac{10}{1296}$ | ? | $\ldots$ | $\frac{1}{1296}$ |

$\rightarrow \quad \mathbb{E}[X] = 4 \times \frac{1}{1296} + 5 \times \frac{4}{1296} + \ldots$
Let $X$ be the sum of 4 fair dice, what is $E[X]$?

<table>
<thead>
<tr>
<th>sum</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>···</th>
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</thead>
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<tr>
<td>$\mathbb{P}[$sum$]$</td>
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<td>$?$</td>
<td>···</td>
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</tr>
</tbody>
</table>

$\Rightarrow \quad E[X] = 4 \times \frac{1}{1296} + 5 \times \frac{4}{1296} + \cdots$

**MUCH** faster to observe that $X$ is a sum,

$$X = X_1 + X_2 + X_3 + X_4,$$

where $X_i$ is the value rolled by die $i$ and

$$E[X_i] = \frac{3}{2}.$$
Let $X$ be the sum of 4 fair dice, what is $\mathbb{E}[X]$?

<table>
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<th>$4$</th>
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Much faster to observe that $X$ is a sum,

$$X = X_1 + X_2 + X_3 + X_4,$$

where $X_i$ is the value rolled by die $i$ and

$$\mathbb{E}[X_i] = 3\frac{1}{2}.$$

Linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + X_3 + X_4] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4]$$

$$= 3\frac{1}{2} + 3\frac{1}{2} + 3\frac{1}{2} + 3\frac{1}{2}$$

$$= 4 \times 3\frac{1}{2} = 14.$$

$\leftarrow$ in general $n \times 3\frac{1}{2}$

**Exercise.** Compute the full PDF for the sum of 4 dice and expected value from the PDF.
Expected Number of Successes in \( n \) Coin Tosses

\( \mathbf{X} \) is the number of successes in \( n \) trials with success probability \( p \) per trial,

\[
\mathbf{X} = \mathbf{X}_1 + \cdots + \mathbf{X}_n
\]

Each \( \mathbf{X}_i \) is a Bernoulli and

\[
\mathbb{E}[\mathbf{X}_i] = p.
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Expected Number of Successes in \( n \) Coin Tosses

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Linearity of expectation,

\[
\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n]
= \mathbb{E}[\mathbf{X}_1] + \mathbb{E}[\mathbf{X}_2] + \cdots + \mathbb{E}[\mathbf{X}_n]
= n \times p.
\]
Expected Waiting Time to \( n \) Successes

\( X \) is the waiting time for \( n \) successes with success probability \( p \).

\[
X = \text{wait to 1st} \quad \bar{X}_1
\]
Expected Waiting Time to $n$ Successes

$X$ is the waiting time for $n$ successes with success probability $p$.

$$X = \text{wait to 1st success} + \text{wait from 1st to 2nd success}$$
**Expected Waiting Time to \( n \) Successes**

\( X \) is the waiting time for \( n \) successes with success probability \( p \).

\[
X = X_1 + X_2 + X_3
\]

where

- \( X_1 \) is the wait to 1st success,
- \( X_2 \) is the wait from 1st to 2nd success,
- \( X_3 \) is the wait from 2nd to 3rd success.

**Expected Value of a Sum: 7 / 12**
Expected Waiting Time to $n$ Successes

$X$ is the waiting time for $n$ successes with success probability $p$.

$$X = \text{wait to 1st} + \text{wait from 1st to 2nd} + \text{wait from 2nd to 3rd} + \cdots + \text{wait from (n-1)th to nth}$$

$$= X_1 + X_2 + X_3 + \cdots + X_n.$$

Each $X_i$ is a waiting time to one success, so

$$\mathbb{E}[X_i] = \frac{1}{p}.$$
Expected Waiting Time to \( n \) Successes

\( \mathbf{X} \) is the waiting time for \( n \) successes with success probability \( p \).

\[
\mathbf{X} = \text{wait to 1st success} + \text{wait from 1st to 2nd success} + \text{wait from 2nd to 3rd success} + \cdots + \text{wait from (n - 1)th to nth success} \\
= \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \cdots + \mathbf{X}_n.
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Linearity of expectation:

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\mathbb{E}[\mathbf{X}] = \mathbb{E}[\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n] \\
= \mathbb{E}[\mathbf{X}_1] + \mathbb{E}[\mathbf{X}_2] + \cdots + \mathbb{E}[\mathbf{X}_n] \\
= \frac{1}{p} + \frac{1}{p} + \cdots + \frac{1}{p} \\
= \frac{n}{p}.
\]

**Example.** If you are waiting for 3 boys, you have to wait 3-times as long as for 1 boy.

**Exercise.** Compute the expected square of the waiting time.
A pack of gum comes with a flag (169 countries). $X$ is the number of gum-purchases to get all the flags.
Coupon Collecting: Collecting the Flags

A pack of gum comes with a flag (169 countries). \( X \) is the number of gum-purchases to get all the flags.

\[
X = \text{wait to } 1st \text{ to get a new flag} \\
X_1 \\
\uparrow \\
p_1 = \frac{n}{n}
\]
Coupon Collecting: Collecting the Flags

A pack of gum comes with a flag (169 countries). $X$ is the number of gum-purchases to get all the flags.

$$X = \text{wait to 1st} + \text{wait from 1st to 2nd}$$

$$X_1 \uparrow \quad \quad \quad X_1 \uparrow$$

$$p_1 = \frac{n}{n} \quad \quad \quad p_2 = \frac{n-1}{n}$$
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$$X = \text{wait to 1st} + \text{wait from 1st to 2nd} + \text{wait from 2nd to 3rd}$$

- $X_1$ (wait to 1st) with $p_1 = \frac{n}{n}$
- $X_1$ (wait from 1st to 2nd) with $p_2 = \frac{n-1}{n}$
- $X_1$ (wait from 2nd to 3rd) with $p_3 = \frac{n-2}{n}$

Expected Value of a Sum: 8 / 12
Coupon Collecting: Collecting the Flags

A pack of gum comes with a flag (169 countries). $X$ is the number of gum-purchases to get all the flags.

$$X = \text{wait to 1st} + \text{wait from 1st to 2nd} + \text{wait from 2nd to 3rd} + \cdots + \text{wait from (n-1)th to nth}$$

$$X_1 \uparrow p_1 = \frac{n}{n}$$

$$X_1 \uparrow p_2 = \frac{n-1}{n}$$

$$X_1 \uparrow p_3 = \frac{n-2}{n}$$

$$X_1 \uparrow p_n = \frac{n-(n-1)}{n}$$
A pack of gum comes with a flag (169 countries). $X$ is the number of gum-purchases to get all the flags.

\[
X = \text{wait to 1st} + \text{wait from 1st to 2nd} + \text{wait from 2nd to 3rd} + \cdots + \text{wait from (n−1)th to nth}
\]

\[
\begin{align*}
X_1 & \uparrow \quad p_1 = \frac{n}{n} \\
X_1 & \uparrow \quad p_2 = \frac{n-1}{n} \\
& \uparrow \quad \vdots \\
X_1 & \uparrow \quad p_n = \frac{n-(n-1)}{n}
\end{align*}
\]

\[
= X_1 + X_2 + X_3 + \cdots + X_n.
\]

$E[X_i] = 1/p_i,$
A pack of gum comes with a flag (169 countries). $\mathbf{X}$ is the number of gum-purchases to get all the flags.

$\mathbf{X} = \text{wait to 1st} + \text{wait from 1st to 2nd} + \text{wait from 2nd to 3rd} + \cdots + \text{wait from (n−1)th to nth}$

$\implies \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \cdots + \mathbf{X}_n.$

$\mathbb{E}[\mathbf{X}_i] = 1/p_i,$

$\mathbb{E}[\mathbf{X}_1] = \frac{n}{n}, \quad \mathbb{E}[\mathbf{X}_2] = \frac{n}{n-1}, \quad \mathbb{E}[\mathbf{X}_3] = \frac{n}{n-2}, \quad \cdots, \quad \mathbb{E}[\mathbf{X}_n] = \frac{n}{n-(n-1)}.$
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\[= X_1 + X_2 + X_3 + \cdots + X_n.\]

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E[X_i] = 1/p_i,
\]

\[
E[X_1] = \frac{n}{n}, \quad E[X_2] = \frac{n}{n-1}, \quad E[X_3] = \frac{n}{n-2}, \quad \cdots, \quad E[X_n] = \frac{n}{n-(n-1)}.
\]

Linearity of expectation:

\[
E[X] = n \left( \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{1} \right) = nH_n \approx n(\ln n + 0.577).
\]

\( n = 169 \rightarrow \) you expect to buy about 965 packs of gum. Lots of chewing!

**Example.** Cereal box contains 1-of-5 cartoon characters. Collect all to get $2 rebate.

Expect to buy about 12 cereal boxes. If a cereal box costs $5, that’s a whopping 3\(\frac{1}{3}\)% discount.
**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.
Iterated Expectation

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An example outcome is $(4; 2, 1, 2, 6)$ with $X_1 = 4$ and $X_2 = 11$: 
**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.

An example outcome is $(4; 2, 1, 2, 6)$ with $X_1 = 4$ and $X_2 = 11$:

$$E[X_2 \mid X_1] = X_1 \times 3\frac{1}{2}.$$ 

The RHS is a function of $X_1$, a random variable.
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Iterated Expectation

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An example outcome is $(4; 2, 1, 2, 6)$ with $X_1 = 4$ and $X_2 = 11$:

$$\mathbb{E}[X_2 \mid X_1] = X_1 \times \frac{3}{2}.$$ 

The RHS is a *function* of $X_1$, a random variable. Compute its expectation.

$$\mathbb{E}[X_2] = \mathbb{E}_{X_1} [\mathbb{E}[X_2 \mid X_1]] \quad \text{(another version of total expectation)}$$
**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.

An example outcome is $(4; 2, 1, 2, 6)$ with $X_1 = 4$ and $X_2 = 11$:

$$
\mathbb{E}[X_2 \mid X_1] = X_1 \times 3^{\frac{1}{2}}.
$$

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$$
\mathbb{E}[X_2] = \mathbb{E}_{X_1}[\mathbb{E}[X_2 \mid X_1]] 
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$$

(another version of total expectation)
Iterated Expectation

**Experiment.** Roll a die and let $X_1$ be the value. Now, roll a second die $X_1$ times and let $X_2$ be the sum of these $X_1$ rolls of the second die.

An example outcome is $(4; 2, 1, 2, 6)$ with $X_1 = 4$ and $X_2 = 11$:

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\mathbb{E}[X_2 \mid X_1] = X_1 \times 3\frac{1}{2}.
$$

The RHS is a *function* of $X_1$, a random variable. Compute its expectation.

$$
\mathbb{E}[X_2] = \mathbb{E}_{X_1}[\mathbb{E}[X_2 \mid X_1]] = \mathbb{E}[X_1] \times 3\frac{1}{2} = 3\frac{1}{2} \times 3\frac{1}{2} = 12\frac{1}{2}.
$$

**Exercise.** Justify this computation using total expectation with 6 cases:

$$
\mathbb{E}[X_2] = \mathbb{E}[X_2 \mid X_1 = 1] \cdot P[X_1 = 1] + \mathbb{E}[X_2 \mid X_1 = 2] \cdot P[X_1 = 2] + \cdots + \mathbb{E}[X_2 \mid X_1 = 6] \cdot P[X_1 = 6].
$$
$W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}].$
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] \].

The first child is either a boy or girl, so by total expectation,

\[
W(k, \ell) = \mathbb{E}[\text{waiting time } | \text{ boy}] \times \frac{p}{1 + W(k-1, \ell)} + \mathbb{E}[\text{waiting time } | \text{ girl}] \times \frac{1-p}{1 + W(k,\ell-1)}
\]

\[
= \frac{1}{1+p} \left( 1 + \frac{1}{1+p} \right) W(k,\ell-1) + \frac{1}{1-p} \left( 1 + \frac{1}{1-p} \right) W(k-1,\ell)
\]

Expected Value of a Sum: 10 / 12

Expected Value of a Product →
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] \]

The first child is either a boy or girl, so by total expectation,

\[
W(k, \ell) = \frac{\mathbb{E}[\text{waiting time} \mid \text{boy}] \times \mathbb{P}[\text{boy}]}{1 + W(k-1, \ell)} + \frac{\mathbb{E}[\text{waiting time} \mid \text{girl}] \times \mathbb{P}[\text{girl}]}{1 - p}
\]

\[
= 1 + pW(k - 1, \ell) + (1 - p)W(k, \ell - 1).
\]
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] . \]

The first child is either a boy or girl, so by total expectation,

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= 1 + pW(k - 1, \ell) + (1 - p)W(k, \ell - 1) .
\]

Base cases: \( W(k, 0) = k/p \) and \( W(0, \ell) = \ell/(1 - p) \)

<table>
<thead>
<tr>
<th>( W(k, \ell) )</th>
<th>( 0 )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( \ell )</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>\cdots</td>
</tr>
<tr>
<td>( \ell )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

\[ W(k, \ell) = \mathbb{E}[\text{waiting time to } k \text{ boys and } \ell \text{ girls}] . \]

The first child is either a boy or girl, so by total expectation,

\[
W(k, \ell) = \frac{\mathbb{E}[\text{waiting time | boy}] \times P[\text{boy}]}{1 + W(k-1, \ell)} + \frac{\mathbb{E}[\text{waiting time | girl}] \times P[\text{girl}]}{1 - p}.
\]

\[
= 1 + pW(k - 1, \ell) + (1 - p)W(k, \ell - 1).
\]

Base cases: \( W(k, 0) = k/p \) and \( W(0, \ell) = \ell/(1 - p) \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( W(k, 0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2 \times p</td>
</tr>
</tbody>
</table>

| \( \ell \) | 1 2 3 4 5 6 7 \cdots |
|---|---|---|---|---|---|---|---|
| 0 | 0 2 4 6 8 10 12 14 \cdots |
| 1 | 2 +1 \times (1-p) |

\[ \mathbb{E}[X] = 12.156 \]

Creator: Malik Magdon-Ismail

Expected Value of a Sum: 10/12

Expected Value of a Product →
Build-Up Expectation: Waiting for 2 Boys and 6 Girls

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\[ = 1 + pW(k - 1, \ell) + (1 - p)W(k, \ell - 1). \]

Base cases: \( W(k, 0) = \frac{k}{p} \) and \( W(0, \ell) = \frac{\ell}{1 - p} \)

<table>
<thead>
<tr>
<th>( W(k, \ell) )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>( \ell )</th>
<th>( \cdot \cdot \cdot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>( \cdot \cdot \cdot )</td>
<td></td>
</tr>
<tr>
<td>( \times p )</td>
<td>( \cdot \cdot \cdot )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3( +1 )</td>
<td>4.5</td>
<td>6.25</td>
<td>8.13</td>
<td>10.06</td>
<td>12.03</td>
<td>14.02</td>
<td>( \cdot \cdot \cdot )</td>
<td></td>
</tr>
<tr>
<td>( \times (1 - p) )</td>
<td>( \cdot \cdot \cdot )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>( \cdot \cdot \cdot )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \cdot \cdot \cdot )</td>
<td>( \cdot \cdot \cdot )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
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\[ \mathbb{E}[X] = 12.156 \]
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<table>
<thead>
<tr>
<th>( W(k, \ell) )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( \ell )</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>( \cdots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W(0, \ell) )</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>( \cdots )</td>
<td></td>
</tr>
<tr>
<td>( W(1, \ell) )</td>
<td>( \times p )</td>
<td>4.5</td>
<td>6.25</td>
<td>8.13</td>
<td>10.06</td>
<td>12.03</td>
<td>14.02</td>
<td>( \cdots )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( W(2, \ell) )</td>
<td>( \times (1-p) )</td>
<td>12.16</td>
<td>14.09</td>
<td>( \cdots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(\mathbb{E}[X] = 12.156\)
Expected Value of a Product

\( X \) is a single die roll:
Expected Value of a Product

\( \mathbf{X} \) is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
\]
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\]

\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X]
\]
Expected Value of a Product

\( X \) is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
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\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X]
\]
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\]

\[
E[X^2] = E[X \times X] = E[X] \times E[X] = (3\frac{1}{2})^2 = 12\frac{1}{4}.
\]
Expected Value of a Product

\( X \) is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
\]

\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X] = \left(3\frac{1}{2}\right)^2 = 12\frac{1}{4}. \times
\]

\( X_1 \) and \( X_2 \) are independent die rolls:

\[
\begin{array}{cccccc}
\hline
\text{Die 1 Value} & 6 & 12 & 18 & 24 & 30 & 36 \\
\text{Die 2 Value} & 5 & 10 & 15 & 20 & 25 & 30 \\
& 4 & 8 & 12 & 16 & 20 & 24 \\
& 3 & 6 & 9 & 12 & 15 & 18 \\
& 2 & 4 & 6 & 8 & 10 & 12 \\
& 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
\end{array}
\]
Expected Value of a Product

\( X \) is a single die roll:

\[
\mathbb{E}[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
\]

\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X] = \left(3\frac{1}{2}\right)^2 = 12\frac{1}{4}.
\]

\( X_1 \) and \( X_2 \) are independent die rolls:

\[
\mathbb{E}[X_1 X_2] = \frac{1}{36} (1+2+\ldots+6+2+4+\ldots+12+3+6+\ldots+18+\ldots+6+12+\ldots+36)
\]
\[
= \frac{441}{36} = 12\frac{1}{4}.
\]
Expected Value of a Product

\( \mathbf{X} \) is a single die roll:

\[
E[X^2] = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6} = 15\frac{1}{6}.
\]

\[
E[X^2] = E[X \times X] = E[X] \times E[X] = \left(3\frac{1}{2}\right)^2 = 12\frac{1}{4}. \times
\]

\( \mathbf{X}_1 \) and \( \mathbf{X}_2 \) are independent die rolls:

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\]

\[
= \frac{441}{36} = 12\frac{1}{4}.
\]

\[
E[X_1 X_2] = E[X_1] \times E[X_2] = \left(3\frac{1}{2}\right)^2 = 12\frac{1}{4}. \checkmark
\]
Expected Value of a Product

\( X \) is a single die roll:

\[
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\]

\[
\mathbb{E}[X^2] = \mathbb{E}[X \times X] = \mathbb{E}[X] \times \mathbb{E}[X] = (3\frac{1}{2})^2 = 12\frac{1}{4}.\times
\]

\( X_1 \) and \( X_2 \) are independent die rolls:

\[
\mathbb{E}[X_1 X_2] = \frac{1}{36}(1+2+\cdots+6+2+4+\cdots+12+3+6+\cdots+18+\cdots+6+12+\cdots+36)
\]

\[
= \frac{441}{36} = 12\frac{1}{4}.
\]

\[
\mathbb{E}[X_1 X_2] = \mathbb{E}[X_1] \times \mathbb{E}[X_2] = (3\frac{1}{2})^2 = 12\frac{1}{4}.\checkmark
\]

---

**Expected value of a product **\( XY \).**

- In general, the expected product is **not** a product of expectations.
- For **independent** random variables, it is: \( \mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y] \).
$X$ is the number of correct hats when 4 hats randomly land on 4 heads.
**Sum of Indicators: Successes in a Random Assignment**

\( X \) is the number of correct hats when 4 hats randomly land on 4 heads.

<table>
<thead>
<tr>
<th>hats:</th>
<th>4</th>
<th>2</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>men:</td>
<td>①</td>
<td>②</td>
<td>③</td>
<td>④</td>
</tr>
</tbody>
</table>

Expected Value of a Sum: 12 / 12
\( X \) is the number of correct hats when 4 hats randomly land on 4 heads.

<table>
<thead>
<tr>
<th>Men</th>
<th>Hat Corrected</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( X_1 = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( X_2 = 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( X_3 = 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( X_4 = 0 )</td>
</tr>
</tbody>
</table>
**Sum of Indicators: Successes in a Random Assignment**

\( X \) is the number of correct hats when 4 hats randomly land on 4 heads.

<table>
<thead>
<tr>
<th>hats:</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>men:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

\[
X = X_1 + X_2 + X_3 + X_4 = 2
\]
\( \mathbf{X} \) is the number of correct hats when 4 hats randomly land on 4 heads.

\[
\begin{array}{c}
\text{hats:} & 4 & 2 & 3 & 1 \\
\text{men:} & 1 & 2 & 3 & 4 \\
\mathbf{X}_1 = 0 & \mathbf{X}_2 = 1 & \mathbf{X}_3 = 1 & \mathbf{X}_4 = 0
\end{array}
\]

\[
\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 + \mathbf{X}_3 + \mathbf{X}_4 = 2
\]

\( \mathbf{X}_i \) are Bernoulli with \( \mathbb{P}[\mathbf{X}_i = 1] = \frac{1}{4} \).
$\mathbf{X}$ is the number of correct hats when 4 hats randomly land on 4 heads.

$$\begin{align*}
\text{hats:} & & \text{men:} \\
\hat{4} & & 1 \\
\hat{2} & & 2 \\
\hat{3} & & 3 \\
\hat{1} & & 4 \\
X_1 &= 0 & X_2 &= 1 & X_3 &= 1 & X_4 &= 0
\end{align*}$$

$$X = X_1 + X_2 + X_3 + X_4 = 2$$

$X_i$ are Bernoulli with $\mathbb{P}[X_i = 1] = \frac{1}{4}$. Linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_4] + \mathbb{E}[X_4] = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 4 \times \frac{1}{4} = 1.$$
**Sum of Indicators: Successes in a Random Assignment**

\( \mathbf{X} \) is the number of correct hats when 4 hats randomly land on 4 heads.

<table>
<thead>
<tr>
<th>hats:</th>
<th>( 4 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>men:</td>
<td>①</td>
<td>②</td>
<td>③</td>
<td>④</td>
</tr>
</tbody>
</table>

\[
X_1 = 0 \quad X_2 = 1 \quad X_3 = 1 \quad X_4 = 0
\]

\[
\mathbf{X} = X_1 + X_2 + X_3 + X_4 = 2
\]

\( \mathbf{X_i} \) are Bernoulli with \( \mathbb{P}[X_i = 1] = \frac{1}{4} \). Linearity of expectation:

\[
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\]

**Exercise.** What about if there are \( n \) people?

**Interesting Example (see text).** Apply sum of indicators to breaking of records.

**Instructive Exercise.** Compute the PDF of \( \mathbf{X} \) and the expectation from the PDF.