1 Basic Definitions of Graph Theory.

Definition 1
An undirected graph $G = (V, E)$ consists of a set $V$ of elements called vertices, and a multiset $E$ (repetition of elements is allowed) of pairs of vertices called edges.

The order of a graph $G(V, E)$ is $|V|$; the size of $G(V, E)$ is $|E|$. A graph of order $p$ and size $q$ is called a $(p, q)$-graph.

If $e = (v, u)$ is an edge in $G$, then $e$ joins $u$ and $v$; $e$ is incident with $u$ and $v$; $u$ and $v$ are endpoints of $e$; $u$ and $v$ are adjacent to each other. The degree of a vertex $v$, denoted $\text{deg}_G(v)$, or $\text{deg}(v)$, is the number of edges of the form $vw$, for some $w$.

A subgraph of a graph $G = (V, E)$ is a graph $H(U, F)$ such that $U \subseteq V$ and $F \subseteq E$; $G$ is a supergraph for $H$. $H$ is an induced subgraph of $G$ if all edges of $G$ connecting vertices in $H$ belong to $H$. $H$ is a spanning subgraph of $G$ if $V(H) = V(G)$. If $G$ is a graph and $S \subseteq V(G)$ then $G - S$ denotes the subgraph of $G$ induced on $V(G) - S$. If $R \subseteq E(G)$, then $G - R$ denotes graph $(V(G), E(G) - R)$.

A walk from a vertex $v$ to a vertex $u$ is a sequence $W$ of alternating vertices and edges

$$v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k,$$

such that $v_1 = v$, $v_k = u$ and $\forall i = 1, \ldots, k - 1$, $e_i$ is incident with $v_i$ and $v_{i+1}$; $k - 1$ is called the length of $W$. A walk without repeated edges is called a trail. A walk (trail) with $v_1 = v_k$ is called closed. If all vertices in $W$ are distinct, $W$ is called a path. If all vertices in a closed walk $W$ are distinct and $k \geq 3$, then $W$ is called a cycle. A graph is called connected if any two vertices $u$ and $v$ are connected by a walk. A subgraph $H$ of $G$ is called a connected component if $H$ is a maximal (non-extensible) connected subgraph of $G$. The number of connected components of a graph $G$ is (often) denoted $k(G)$.

A distance $\text{dist}(v, u)$ between two vertices $u$ and $v$ of a connected graph is the length of the shortest path connecting them. For a connected graph $G$,

$$E(v) = \max_{x \in V(G)} \text{dist}(v, x)$$

the eccentricity of $v$ in $G$;

$$D(G) = \max_v E(v)$$

the diameter of a $G$;

$$R(G) = \min_v E(v)$$

the radius of $G$.

A graph is called a tree, if it is connected and has no cycles. A leaf is a vertex of degree 1. A rooted tree is a tree with a distinguished vertex, called root. A level of a vertex
in a rooted tree is the distance from the vertex to the root. If a vertex \( v \) lies on the path from the root to a vertex \( u \), then \( v \) is called an ancestor of \( u \); if additionally, \( u \) and \( v \) are adjacent then \( u \) is a child of \( v \) and \( v \) is a parent of \( u \). An ordered tree is a rooted tree in which the children of every vertex are assigned a fixed ordering. A binary tree is an ordered tree in which every vertex has at most two children, one of which is called the left child and the other is called the right child.

**Theorem 1** *Every tree has at least two leaves.*

**Proof.** Consider a longest path \( P \) in a given tree \( T \) (note that there can be more than one longest path). Let \( u \) and \( v \) be the endpoints of \( P \). Then each of these two vertices is a leaf, since otherwise either, \( T \) has a cycle, or \( P \) is not a longest path. \( \square \)

**Theorem 2** *A tree with \( p \) vertices has \( q = p - 1 \) edges.*

**Proof.** We prove the statement by induction on \( p \).

**BASE.** For \( p = 1 \), every graph \( T \) has \( 0 = p - 1 \) edges.

**INDUCTIVE STEP.** Let the statement hold for some \( p \geq 0 \) and let \( T \) be a tree with \( p + 1 \) vertices. Then by Theorem 1, \( T \) has a leaf \( x \). \( T - x \) must be a tree, since if \( x \) were on a path connecting some two vertices then \( x \) would be of degree \( \geq 2 \). In the tree \( T - x \), there are \( p - 1 \) vertices and \( p - 2 \) edges (by induction.) We finish the proof noticing that removing \( x \), removes exactly one edge. \( \square \)

2 **Digraphs: Basic Definitions**

A directed graph, or digraph \( G = (V, E) \) consists of \( V \): a non-empty set of elements called vertices, or nodes, and \( E \): a set of pairs of ordered vertices called edges, or arcs.

Example: \( V = \{ a, b, c, d \} \); \( E = \{(a, b)(a, c)(d, b)\} \)

\[ \begin{array}{cccc}
  a & b & c & d \\
  \end{array} \]

The order (size) of a graph \( G(V, E) \) is the number of vertices (edges) or vertex number (edge number), written \( |V| (|E|) \). A graph of order \( p \) and size \( q \) is called a \((p,q)\)-graph.

If \( e = (v, u) \) is an edge in \( G \), then
For a digraph $G$ parallel arcs (same tail and same head) are not allowed; loops (an arc whose tail coincides with the head) are not allowed either. If parallel edges and/or loops are allowed, the object is called **directed pseudo-graph**.

For a digraph $G$:

- $N^+(v) = \{u : (v, u) \in E(G)\}$; out-neighborhood;
- $N^-(v) = \{u : (u, v) \in E(G)\}$; in-neighborhood;
- $N^+(v) \cup N^-(v) = \{u : (u, v) \in E(G) \text{ or } (v, u) \in E(G)\}$; neighborhood;
- If $A \subseteq V(G)$, then $N^+ (A) = \bigcup_{x \in A} N^+(x) - A$.
- $N^-(A) = \bigcup_{x \in A} N^-(x) - A$.

- $\deg^+(v) = |N^+(v)|$ is the **out-degree** of $v$;

- $\deg^-(v) = |N^-(v)|$ is the **in-degree** of $v$;

- $\deg(v) = \deg^+(v) + \deg^-(v)$ is the **degree** of $v$;

- $\delta^+(G) = \min\{\deg^+(v)\}$; $\Delta^+(G) = \max\{\deg^+(v)\}$;

- $\delta^-(G) = \min\{\deg^-(v)\}$; $\Delta^-(G) = \max\{\deg^-(v)\}$;

A graph is called **regular** if

$$\min\{\delta^+(G), \delta^-(G)\} = \max\{\Delta^+(G), \Delta^-(G)\}.$$ 

**Theorem 3**

$$\sum_{x \in V(G)} \deg^-(x) = \sum_{x \in V(G)} \deg^+(x) = |E(G)|.$$

A digraph $H$ is a **subgraph**, or **subdigraph** of $G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. $G$ is called a **supergraph**, or **superdigraph** for $H$.

$H$ is a **spanning** subgraph of $G$, if $V(H) = V(G)$. $H$ is an **induced** subgraph of $G$, if every edge $(v, u) \in E(G)$ for which $u, v \in V(H)$, is also an edge in $H$.

**Reversing** the edge $(u, v)$ is replacing the edge with $(v, u)$. The **converse** of $G$ is a digraph obtained from $G$ by reversing all edges.
Contraction of a subgraph $H$ in a graph $G$ is the replacement of $V(H)$ with a new vertex $z$ and creating new edges $(x, z)$ (resp. $(z, y)$) iff there is any edge $(x, t)$ (resp. $(s, y)$), where $t \in V(H)$ (resp. $s \in V(H)$).

The underlying graph of $G = (V, E)$ is an undirected graph $K = (V, E')$, where $(x, y) \in E'$ iff either $(x, y) \in E$, or $(y, x) \in E$.

A walk of a graph $G = (V, E)$ is a an alternating sequence

$$W = x_1a_1x_2 \cdots x_{k-1}a_{k-1}x_k,$$

where $\forall i, x_i \in V(G)$ and $(x_i, x_{i+1}) \in E(G)$.

A trail is a walk in which all edges are distinct. A path is a trail in which all vertices are distinct. A cycle is a closed walk in which all vertices are distinct, except the last and the first.

A vertex $x$ is reachable from $y$, if $G$ has an $(y, x)$-walk. The digraph is strongly connected, or strong, if every vertex is reachable from any other vertex. A digraph is called weakly-connected, if its underlying graph is connected.

**Proposition 1** If digraph $G$ has an $x, y$-walk, then it has an $x, y$-path.

A digraph $G$ is called acyclic, or a DAG, if it has no cycle. An ordering $x_1, x_2, \ldots, x_n$ of $G$ is called acyclic, if no edge $(x_i, x_j)$ exists in $G$ with $i > j$.

**Proposition 2** Every acyclic digraph $G$ has an acyclic ordering $x_1, x_2, \ldots, x_n$; the in-degree of $x_1$ is zero and the out-degree of $x_n$ is zero.

**Proposition 3** If $G$ is an acyclic graph with exactly one vertex $x$ of in-degree zero, and exactly one vertex $y$ of out-degree zero, then for every $v \in V(G)$, there is an $(x, v)$-path and $(v, y)$-path.

A maximal strong subgraph of a given digraph $G$ is called a strongly connected component, or a strong component of $G$. 


Proposition 4 Let $G$ be a digraph and let $H_1, H_2$ be two distinct strong components of $G$. Then, $V(H_1)$ and $V(H_2)$ are disjoint sets. If $x \in V(H_2)$ is reachable from $y \in V(H_1)$, then no vertex from $H_1$ is reachable from any vertex in $H_2$.

Let $H_1, H_2, \ldots, H_k$ be the set of all strong components of a digraph $G$. A new digraph $F = SC(G)$ is defined whose vertices are $z_1, z_2, \ldots, z_k$ and $(z_i, z_j) \in E(F)$ iff $G$ has an edge from $H_i$ to $H_j$.

Proposition 5 For every digraph $G$, $SC(G)$ is an acyclic digraph.

A strong component of a graph $G$ which corresponds to a vertex $x$ in $SC(G)$ of in-degree zero, is called an **initial strong component**.
3 Graph representation

Adjacency matrix:

\[
A = \begin{pmatrix}
    a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\
    a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n,1} & a_{n,2} & \ldots & a_{n,n}
\end{pmatrix}; \quad a_{i,j} = \begin{cases} 
    1, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\
    0, & \text{otherwise.}
\end{cases}
\]

Incidence matrix:

\[
A = \begin{pmatrix}
    b_{1,1} & b_{1,2} & \ldots & b_{1,q} \\
    b_{2,1} & b_{2,2} & \ldots & b_{2,q} \\
    \vdots & \vdots & \ddots & \vdots \\
    b_{p,1} & b_{p,2} & \ldots & b_{p,q}
\end{pmatrix}; \quad a_{i,j} = \begin{cases} 
    1, & \text{if vertex } v_i \text{ incident with edge } e_j, \\
    0, & \text{otherwise.}
\end{cases}
\]

Adjacency list:

4 Distances in digraphs

The distance \( \text{dist}(x, y) \) from \( x \) to \( y \) is the length of the shortest \((x, y)\)-path.

Define

\[
\text{dist}(X, Y) = \max_{x \in X; y \in Y} \text{dist}(x, y);
\]
\[
\begin{align*}
\text{rad}^+(G) &= \min_v \text{dist}(v, V(G)); \\
\text{rad}^-(G) &= \min_v \text{dist}(V(G), v); \\
\text{diam}(G) &= \max_v \max_u \text{dist}(v, x); \\
\text{rad}(G) &= \min_v \frac{\text{dist}(v, V) + \text{dist}(V, x)}{2}.
\end{align*}
\]

**Theorem 4** Let \( A \) be the adjacency matrix of a digraph \( G \) with \( n \) vertices and let \( A^k \) be the \( k \)-th power of \( A \). Then for every \( i, j \in [0, n - 1] \), the entry \( a_{i,j}^k \) of \( A^k \) is the number of walks of length \( k \) from \( v_i \) to \( v_j \).

**Proof.** Induction on \( k \). \( \blacksquare \)
5 Set-Theory.

Definition 2 Given two sets $A$ and $B$, a function $f : A \to B$ is called one-to-one, if
\[ \forall x, y \in A, x \neq y, f(x) \neq f(y). \]

Definition 3 Given two sets $A$ and $B$, a function $f : A \to B$ is called onto, if
\[ \forall y \in B, \exists x \in A, f(x) = y. \]

Definition 4 Two sets $A$ and $B$ are called of the same size, or of the same cardinality, if there exists a one-to-one, onto function $f : A \to B$.

Definition 5 If a set $A$ is of the same cardinality as $\mathcal{N}$, it is called countable; otherwise, the set is called uncountable.

Proposition 6
Let $\mathcal{N}$ be the set of all non-negative integers, and let $\mathcal{N}_e$ be the set of all even non-negative integers. Then $\mathcal{N}$ and $\mathcal{N}_e$ are of the same cardinality.

Proof. Consider function $f : \mathcal{N} \to \mathcal{N}_e$ defined by
\[ f(i) = 2i, \ i \in A. \]

Definition 5 If a set $A$ is of the same cardinality as $\mathcal{N}$, it is called countable; otherwise, the set is called uncountable.
**Theorem 5** Let $\mathcal{Q}$ be the set of positive rational numbers. Then $\mathcal{Q}$ is countable.

**Proof.** We consider a set $\mathcal{Q}'$, comprised of all rationals $p/q$, where $p, q = 1, 2, 3, \ldots$. Clearly, every positive rational number $r$ is repeated in $\mathcal{Q}'$ an infinite number of times. We prove that even $\mathcal{Q}'$ is countable. The correspondence $f$ is defined by the formula

$$f(\frac{p}{q}) = \frac{(p + q - 1)(p + q - 2)}{2} + p$$

and is illustrated in the figure below.

![Diagram showing the correspondence](image)

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**Theorem 6** Let $\text{SUB}_f$ be the set of all finite subsets of $\mathcal{N}$. Then $\text{SUB}_f$ is countable.

**Theorem 7** The set $\mathcal{R}$ of all infinite 0,1-sequences is not countable.

**Proof.** Suppose that contrary to the claim, $\mathcal{R}$ is countable and let $f : \mathcal{N} \to \mathcal{R}$ be a one-to-one correspondence between these sets. Denote $f_i(k)$ to be the $i$th entry of the sequence $f(k)$ ($i, k > 0$). Now, consider the sequence

$$s = \{1 - f_1(1), 1 - f_2(2), 1 - f_3(3), \ldots, 1 - f_k(k), \ldots\}.$$ 

By construction, the $k$th entry of $s$ is different from $f_k(k)$, thus $s$ is different from every $f(k)$ ($k = 1, 2, \ldots$). This proves that $f$ is not a mapping onto; consequently, $\mathcal{R}$ is not countable. 

**Remark 1** The argument used for the proof of the previous theorem is called the **diagonalization** argument.