1 Matchings in Graphs
Definition 1

Two edges are called **independent** if they are not adjacent in the graph. A set of mutually independent edges is called a matching.

A matching is called

- **maximal** if no other matching contains it.
- **maximum** if its cardinality is maximal among all matchings
- **perfect** if every vertex of the graph is incident to an edge of the matching.

**maximal vs maximum**

![Diagram showing maximal and maximum matchings](image)

How to construct a maximal matching?
Given a matching $M$ in $G$,
a path is called $M$-alternating if its edges are alternatively in $M$
and not in $M$.
a vertex $x$ is called weak if no edge of $M$ is incident to $x$.

Using alternating path connecting weak vertices
The symmetric difference of two sets $X$ and $Y$ is the set of all elements that belong to one but not the other of the sets.

$$X \otimes Y = (X \cup Y) - (X \cap Y)$$

**Theorem 1** A matching $M$ is maximum iff there exists no alternating path between any two distinct weak vertices of $G$.

**Proof.** For a proof, we will try to answer the following questions:

- If a graph $M$ is a matching, what is the maximum degree of a vertex in such a graph?

- If the edge set $E$ of a graph $F$ is the symmetric difference of two matchings $M_1$ and $M_2$, then what is the maximal vertex degree of $F$?

- Consider a component $C$ of $F$ above. Can $C$ be a path, a cycle, anything else?

- If $C$ is a cycle of $F$, can it have an odd length; an even length?

**Conclusion:** Let $M$ be a matching which is not maximum and let $M^*$ be maximum. Then at least one connected component of $M \otimes M^*$ is an alternating path containing more edges from $M^*$.

That component is an augmenting path for $M$. \qed
Definition 2
An edge cover of graph $G$ is a set $L$ of edges such that every vertex is an endpoint of an edge in $L$.

$\beta'(G)$ is the smallest size of an edge cover.

$\alpha'(G)$ is the largest size of a matching in $G$.

Theorem 2
For every graph $G$ without isolated vertices,

$$\alpha'(G) + \beta'(G) = n(G).$$

Proof.
$\beta'(G) \leq n(G) - \alpha'(G)$. Starting with a matching, add one edge for every unsaturated vertex. The total is $\alpha'(G) + n - 2\alpha'(G) = n - \alpha'(G)$, a cover size.

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\]

maximum matching unsaturated vertices

$\alpha'(G) \geq n(G) - \beta'(G)$. Consider the minimal edge cover $L$ and the subgraph $H = (V, L)$.

$H$ doesn’t have paths of length $> 2$, since otherwise an edge inside could be removed yielding a smaller edge cover. Thus, every connected component of $H$ is a star.

If $k$ is the number of these stars, $|L| = n(G) - k$. Form a matching by picking one edge from each star.
2 Matchings in Bipartite Graphs

Definition 3

A graph $G(V, E)$ is called \textbf{bipartite} if $V$ can be partitioned into two subsets $V_1$ and $V_2$ so that $E \subseteq V_1 \times V_2$.

**Proposition 1** A graph $G$ is bipartite if and only if it has no cycles of odd length.

**Proof.** If $G$ is bipartite and $C$ is a cycle of $G$ then the vertices of $C$ can be labeled by 0 and 1 depending to which part the vertex belongs to. This implies that the length of the cycle is even.

Let now $G$ be a graph without cycles of odd length. We want to prove that $G$ is bipartite. Obviously, we may assume that $G$ is connected. Then, consider an arbitrary vertex $v \in V(G)$ and define $V_{\text{even}}$ (resp. $V_{\text{odd}}$) to be the set of all vertices whose distance from $v$ is even (resp. odd). If there were an edge connecting two vertices from $A_{\text{even}}$ (resp. two vertices from $A_{\text{odd}}$), then $G$ would have an odd cycle. Thus, $G$ is bipartite and $(V_{\text{even}}, V_{\text{odd}})$ is the partition. \hfill \Box

Definition 4

Given a matching $M$ of a bipartite graph $G = (V_1, V_2; E)$, $V_1(M)$ (resp. $V_2(M)$) denotes the set of vertices in $V_1$ (resp. $V_2$) incident to the edges in $M$. 

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Definition 5

A matching $M$ is said to **saturate** a vertex $v$, if there is an edge $(v, w) \in M$.

**Theorem 3** A bipartite graph $G = (V_1, V_2, E)$ contains a matching with $|V_1|$ edges (a matching saturating all $V_1$) iff

$$\forall X \subseteq V_1, \ |X| \leq |E(X)|.$$  \hfill (\ast)

**Proof.** First, we prove that the existence of a matching saturating all $V_1$ implies $|X| \leq |E(X)|$.

Indeed, since $M$ is a matching, the sets $V_1(M)$ and $V_2(M)$ have the same cardinality. For every $x \in X$ there is a distinct vertex $y \in E(X)$ for which $(x, y)$ is in $M$.

Now we prove that condition

$$\forall X \subseteq V_1, \ |X| \leq |E(X)|$$ \hfill (\ast)

implies the existence of a matching which saturates all $V_1$.

Applying (\ast) to $X = V_1$, we get $|V_1| \leq |V_2|$. Our goal is to prove that there is a matching which saturates all vertices in $V_1$.

Suppose $M_0$ is a maximum matching and it doesn’t saturates all $V_1$. We will show that this supposition leads to a contradiction to the condition (\ast). The idea is illustrated in the next figure.
Let $S = V_1 - V_1(M_0)$ and $T = V_2 - V_2(M_0)$. If there were an edge $(x, y) \in E$ such that $x \in S$ and $y \in T$, the new matching is $M_0 \cup \{(x, y)\}$ contrary to the maximality of $M_0$.

So, we assume that no vertex in $S$ is adjacent to any vertex in $T$.

Because of (*), for every vertex $v \in S$, there are edges incident to $v$. Consider any path in $G$ satisfying the following conditions:

1. the first vertex of the path is in $S$;

2. the edges of the path alternate between edges not in $M_0$ and edges in $M_0$.

Claim. No path satisfying (1) and (2) contains a vertex in $T$.

Indeed, if such a path had a vertex in $T$, it would be its last vertex, and the path itself would be an $M_0$-alternating path with two end-vertices that are not saturated by $M_0$. Thus, the path would have been augmenting, yielding a matching $M_1$ which is larger than $M_0$ contrary to the maximality of $M_0$. 

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S \\
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\]
Let \( v \in S \) and \( D \) be the set of all vertices reachable from \( v \) by \( M_0 \)-alternating paths.

Because of the Claim, \( D \cap T = \emptyset \). Furthermore,

\[
|V_1(M_0) \cap D| = |V_2(M_0) \cap D|.
\]

Thus, for \( X = |V_1(M_0) \cap D| \cup \{v\} \),

\[
|E(X)| = |V_2(M_0) \cap D|,
\]

which implies that

\[
|E(X)| = |X| - 1.
\]

This contradicts the condition on \( G \) and thus proves that \( S = \emptyset \). \( \blacksquare \)
Definition 6

Given a graph $G(V, E)$, a subset $C \subseteq V$ is called \textbf{vertex cover}, if every edge in $E$ is incident to at least one vertex in $C$.

\textbf{Question:} if $C$ is a vertex cover for $G$, what kind of graph will be obtained if $C$ is removed from $G$?

\textbf{Theorem 4} (Egervári (1931); König (1931)) A maximum cardinality of a matching in a bipartite graph $G = (V_1, V_2, E)$ is equal to the minimum cardinality of a vertex cover.

\textbf{Proof.}

⇒ Let $c$ be the smallest size of a vertex cover and $m$ be the largest size of a matching in $G$. Then $c \geq m$, since in any cover, every edge in a matching must be “covered” by its own vertex.

⇐ Given a minimum vertex cover $C$, construct a matching of size $|C|$, which proves that $m \geq c$. 
Split the graph into two subgraphs using the minimum vertex cover

The left subgraph (resp. right) satisfies the condition

$$|X| \leq |E(X)|$$

The left subgraph (resp. right) must have a matching which saturates partition $C^\wedge V_1$ (resp. $C^\wedge V_2$)