1 Plane and Planar Graphs

**Definition 1**
A graph $G(V, E)$ is called *plane* if

- $V$ is a set of points in the plane;
- $E$ is a set of curves in the plane such that
  1. every curve contains at most two vertices and these vertices are the ends of the curve;
  2. the intersection of every two curves is either empty, or one, or two vertices of the graph.

**Definition 2**
A graph is called *planar*, if it is isomorphic to a plane graph. The plane graph which is isomorphic to a given planar graph $G$ is said to be *embedded* in the plane. A plane graph isomorphic to $G$ is called its *drawing*. 

$G$ is a planar graph $H$ is a plane graph isomorphic to $G$
The adjacency list of graph $F$. Is it planar?

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<tr>
<td>1</td>
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<td>1 3 8 7</td>
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<td>12</td>
<td>9 5 4 6</td>
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What happens if we add edge (1,12)? Or edge (7,4)?
Definition 3

A set $U$ in a plane is called **open**, if for every $x \in U$, all points within some distance $r$ from $x$ belong to $U$.

A **region** is an open set $U$ which contains polygonal $u, v$-curve for every two points $u, v \in U$.

![area without the boundary](image)

Definition 4

Given a plane graph $G(V, E)$, a **face** of $G$ is a maximal region of the plane if the vertices and the edges of $G$ are removed.

An unbounded (infinite) face of $G$ is called **exterior**, or outer face.

The vertices and the edges of $G$ that are incident with a face $F$ form the **boundary** of $F$.

Proposition 1

In a plane graph, every cycle is the symmetric difference of the boundaries of some faces.
**Proposition 2**

For every face of a given plane graph $G$, there is a drawing of $G$ for which the face is exterior.
Dual Plane Graphs

Definition 5

Let $G$ be a plane graph. The dual graph $G^*$ of $G$ is a new plane graph having a vertex for each face in $G$ and the edges that correspond to the edges of $G$ in the following manner:

- if $e$ is an edge of $G$ which separates two faces $X$ and $Y$, then the corresponding dual edge $e^* \in E(G^*)$ is an edge joining the vertices $x$ and $y$ that correspond to $X$ and $Y$ respectively.

Remark. Different plane drawings (embeddings) of the same planar graph may have non-isomorphic duals.

Construct duals to these drawings.
**Definition 6**

The **length** of a face \( F \) in a plane graph \( G \) is the total number of edges in the closed walks in \( G \) that bound the face.

**Proposition 3**

If \( l(F_i) \) denotes the length *(the number of edges in its boundary)*, of face \( F_i \) in a plane graph \( G \), then

\[
2e(G) = \sum l(F_i).
\]

**Definition 7**

A bond of a graph \( G \) is a **minimal non-empty edge cut**.

**Proposition 4**

Edges of a plane graph \( G \) form a cycle iff the corresponding edges in \( G^* \) form a bond.
**Theorem 1 (Euler):**

If $G$ is a connected plane $(p, q)$-graph with $r$ faces, then

$$p - q + r = 2.$$  

**Proof.** We prove it by induction on $q$.

**Base.** If $q = 0$, then $p = 1$; obviously, $r = 1$, and the result follows.

**Inductive Step.** Assume that the Euler Theorem holds true for all connected graphs with fewer than $q$ ($q \geq 1$) edges, and let $G$ be a connected plane graph with $q$ edges.

If $G$ is a tree, then $p = q + 1$ and $r = 1$ (the only face is exterior) implying the result.

If $G$ is not a tree, then it has an enclosed face. The edges of the face form a cycle. Take any edge $e$ on the cycle and consider graph

$$H = G - e.$$  

Since $q(H) = q - 1$, by induction, $p(H) - q(H) + r(H) = 2$. But $p(H) = p$ and $r(H) = r - 1$. The result follows. 

**Theorem 2**

If $G$ is a planar graph (no parallel edges) with $p$ vertices and $q$ edges, $(q \geq 3)$, then $q \leq 3p - 6$. If, in addition, $G$ is bipartite, then $q \leq 2p - 4$.

**Proof.** Let $r$ be the number of faces of $G$ and let $m_i$ be the number of edges in the boundary of the $i^{th}$ face ($i = 1, \ldots, r$).

Since every face contains at least three edges,

$$3r \leq \sum_{i=1}^{r} m_i.$$  

On the other hand, since every edge can be in the boundary of at most two faces,

$$\sum_{i=1}^{r} m_i \leq 2q.$$  

Thus, $3r \leq 2q$ and by Euler’s Theorem, $p - q + 2q/3 \geq 2$, implying $q \leq 3p - 6$.

If $G$ is bipartite, the shortest cycle is of length at least 4. Thus,

$$4r \leq \sum_{i=1}^{r} m_i.$$  

Together with $\sum_{i=1}^{r} m_i \leq 2q$ and $p - q + r = 2$, we get the second part of the Theorem.
Corollary 1 Every planar graph $G$ contains a vertex of degree at most 5.

Proof. (HINT: assuming that all degrees are $\geq 6$, estimate the number of edges in the graph, and compare your estimate with that by Theorem 2.)

Corollary 2 Graphs $K_{3,3}$ and $K_5$ are not planar.

Proof. For $K_5$, $p = 5$ and $q = 10$. Thus, $q > 3p - 6 = 9$ and by Theorem 2, $K_5$ is not planar.
If $K_{3,3}$ were planar, then the second part of Theorem 2 would apply, leading to a contradiction, since for this graph $p = 6$, $q = 9$, and $q > 2p - 4$.

Definition 8 A subdivision of an edge $ab$ in a graph $G$ is an operation which replaces $ab$ with two edges $az$ and $zb$ where $z$ is a new vertex different from other vertices of $G$. The result of the subdivision is also called a subdivision of $G$.

A Kuratowski graph is a graph obtained by several subdivisions from either $K_5$ or $K_{3,3}$.

Corollary 3 No Kuratowski graph is planar.
2 Coloring Planar Graphs

\textbf{Theorem 3} Every planar graph $G$ is 5-colorable.

\textbf{Proof.} By induction on the number $n(G)$ of vertices.

\textbf{Base.} For all planar graphs with $n(G) \leq 5$, the statement is correct.

\textbf{Inductive step.} Let $G$ have more than 5 vertices. Select a vertex $v$ of degree $\leq 5$. It always exists, since else, the number of edges in the graph would exceed the upper bound of $3p - 6$. By induction, graph $G - v$ is 5-colorable.

Consider a 5-coloring of $G - v$. If any color, 1 2, 3, 4, 5 is not used for vertices adjacent to $v$, use it for $v$. Thus, we need to assume
that \( v \) has 5 neighbors that are colored using all 5 colors. Let us call those neighbors according to their colors: \( v_1, v_2, v_3, v_4, v_5 \) (see Figure below).

Consider a bipartite graph \( H \) induced by all vertices of \( G - v \) whose colors are 1 or 3, and let \( C \) be the connected component of \( H \) which contains vertex \( v_1 \). If \( C \) does not contain vertex \( v_3 \), then we re-color \( C \): every vertex of \( C \) whose color is 1 (resp. 3) gets color 3 (resp. 1). This recoloring frees color 1 for \( v \), yielding a 5-coloring of \( G \).

Finally, assume that \( C \) contains vertex \( v_3 \). Thus, there is a 2-colored path connecting vertices 1 and 3. This path together with vertex \( v \) forms a cycle which makes it impossible the existence of a path colored 2 and 4 connecting vertices \( v_2 \) and \( v_4 \). Thus, recoloring the 2-colored connected component containing vertex \( v_2 \) makes color 2 available for coloring \( v \).
From colorings faces of a map to coloring its edges.

The coloring of the faces of an arbitrary map is reduced to that of a map for which every vertex has three incident edges. See the Figure below.

An edge coloring of a graph $G$ is an assignment of colors to edges such that any two edges incident to the same vertex have different colors.
**Theorem.** (Tait, 1878). A 3-regular planar graph without bridges is 4-face colorable iff it is 3-edge-colorable.

**Proof.** Suppose $G$ satisfies the conditions of the theorem and it is 4-face-colorable. Let the colors be $c_0 = 00$, $c_1 = 01$, $c_2 = 10$, $c_3 = 11$. Because $G$ is bridge-less, every edge bounds two distinct faces. Given an edge between faces colored $c_i$ and $c_j$, assign this edge the color obtained by adding $c_i$ and $c_j$ coordinate-wise modulo 2. For example: $c_0 + c_1 = c_1; \ c_1 + c_3 = c_2; \ c_2 + c_3 = c_1$.

**The reverse.** Suppose now that $G$ has a proper 3-edge-coloring, and let the colors be 1, 2, and 3. Since $G$ is 3-regular, every color appears at every vertex. Let $E_1, E_2,$ and $E_3$ be the edge-sets colored 1, 2, and 3, respectively. The union of any two of them is the union of disjoint cycles. Let $H_1 = E_1 \cup E_2$ and $H_2 = E_1 \cup E_3$.

The faces of graph $H_1$ can be 2-colored with two colors $\alpha$ and $\beta$.

The faces of graph $H_2$ can be 2-colored with two colors $\gamma$ and $\delta$.

Then each face of $G$ is a subset of a face in $H_1$ and a subset of a face in $H_2$. Assign to each face in $G$ a pair of colors $(x, y)$ where $x \in \{\alpha, \beta\}$ and $y \in \{\gamma, \delta\}$.

It is easy to prove that any two adjacent faces of $G$ get pairs off colors
that are differ in at least one coordinate. Thus the assignment is a 4-coloring of the faces of $G$.

**Tutte’s conjecture:** every bridge-less 3-regular graph $G$ which is not three-edge colorable has the Petersen graph as its minor, that is the Petersen graph can be obtained from $G$ by contracting and removing edges.
Problem 1 Use the fact that every cycle of the Petersen graph is of length 5 or more to prove that it is non-planar.

Problem 2 Show that the Heawood graph below is non-planar.

Problem 3 Prove that a plane graph $G$ is bipartite iff its dual $G^*$ is Eulerian.

Definition 9

A subdivision of an edge $e = (xy)$ is an operation which removes $e$ from the graph, adds a new vertex $z$ along with two new edges: $(xz)$ and $(zy)$. A subdivision of a graph $G$ is a graph that can be obtained from $G$ by a sequence of edge subdivisions.
**Definition 10**

Two graphs $G$ and $H$ are called *homeomorphic* if either $G \approx H$, or there is a graph $F$ such that both $G$ and $H$ can be obtained from $F$ by two separate subdivisions of $F$.

![Diagram of homeomorphic graphs](image)

**Definition 11** The operation of **contraction** of an edge $e = xy$ of a graph $G$ removes from the graph $e$, $x$, and $y$, and adds a new vertex $z$, which is adjacent to any old vertex $u$ iff $u$ is adjacent to at least one of $x$ or $y$. The resulting graph is denoted $G/(x,y)$.

![Diagram of contraction operation](image)
Problem 4 If $G$ is non-planar, then every subdivision of $G$ is non-planar.

Problem 5 If $G$ is planar, then every contraction of $G$ is planar.

Problem 6 If the contraction $G/(x,y)$ has a subgraph homeomorphic to $K_5$ or $K_{3,3}$, then $G$ has a subgraph homeomorphic to either $K_5$ or $K_{3,3}$.

Definition 12 A maximal planar graph is a simple planar graph which is not a spanning subgraph of another planar graph. A triangulation is a simple plane graph where every face is a 3-cycle.

Problem 7 For a simple plane graph $G$ with $n$ vertices, the following are equivalent

1. $G$ has $3n - 6$ edges;
2. $G$ is a triangulation; and
3. $G$ is a maximal planar graph.

Problem 8 Give two examples of planar graphs with no vertex of degree less than five.

Problem 9 Show that every planar graph of order $\geq 4$ has at least four vertices of degree less than or equal to 5.
Problem 10 Show that $K_5 - e$ is planar for any edge $e \in E(K_5)$. Show that $K_{3,3} - e$ is planar for any edge $e \in E(K_{3,3})$.

Problem 11 A graph is called outerplanar, if it can be embedded into a plane so that every vertex of the graph lies on the boundary of then exterior face. Show that a planar $G$ is outerplanar if and only if $G + K_1$ is planar.