1 Plane and Planar Graphs

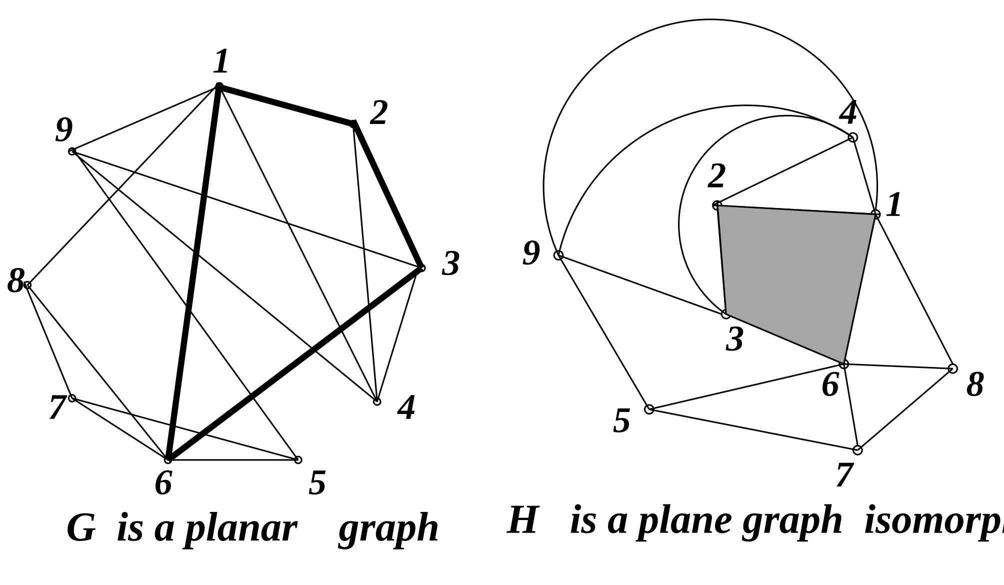
Definition 1

A graph $G(V, E)$ is called *plane* if

- V is a set of points in the plane;
- E is a set of curves in the plane such that
  1. every curve contains at most two vertices and these vertices are the ends of the curve;
  2. the intersection of every two curves is either empty, or one, or two vertices of the graph.

Definition 2

A graph is called *planar*, if it is isomorphic to a plane graph. The plane graph which is isomorphic to a given planar graph G is said to be **embedded** in the plane. A plane graph isomorphic to G is called its **drawing.**



G* is a planar graph** ***H* is a plane graph isomorphic to *G

The adjacency list of graph F. Is it planar?

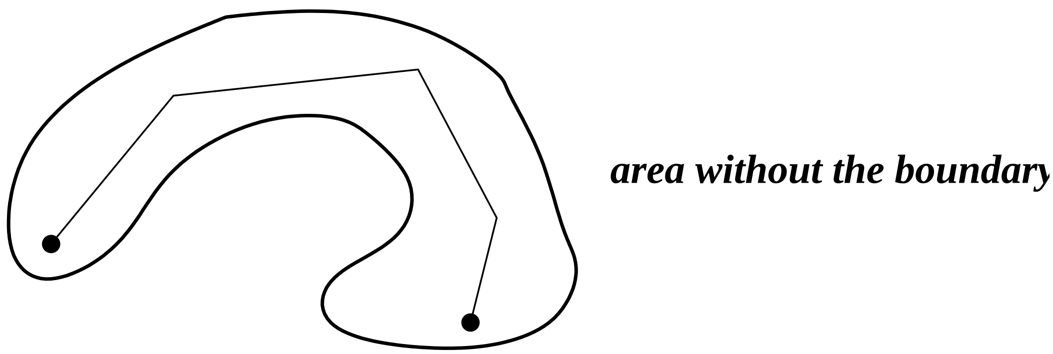
1	4 5 6 8 9 11
2	9 7 6 10 3
3	7 11 8 2
4	1 5 9 12
5	1 12 4
6	1 2 8 10 12
7	2 3 9 11
8	1 11 3 6 10
9	7 4 12 1 2
10	2 6 8
11	1 3 8 7
12	9 5 4 6

What happens if we add edge (1,12)? Or edge (7,4)?

Definition 3

A set U in a plane is called **open**, if for every $x \in U$, all points within some distance r from x belong to U.

A **region** is an open set U which contains polygonal u, v-curve for every two points $u, v \in U$.



Definition 4

Given a plane graph $G(V, E)$, a **face** of G is a maximal region of the plane if the vertices and the edges of G are removed.

An unbounded (infinite) face of G is called **exterior**, or outer face.

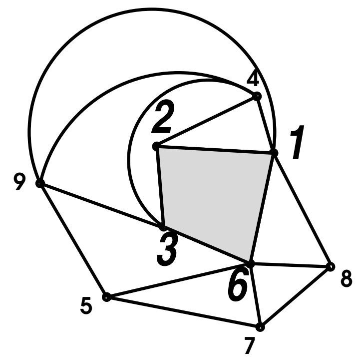
The vertices and the edges of G that are incident with a face F form the **boundary** of F.

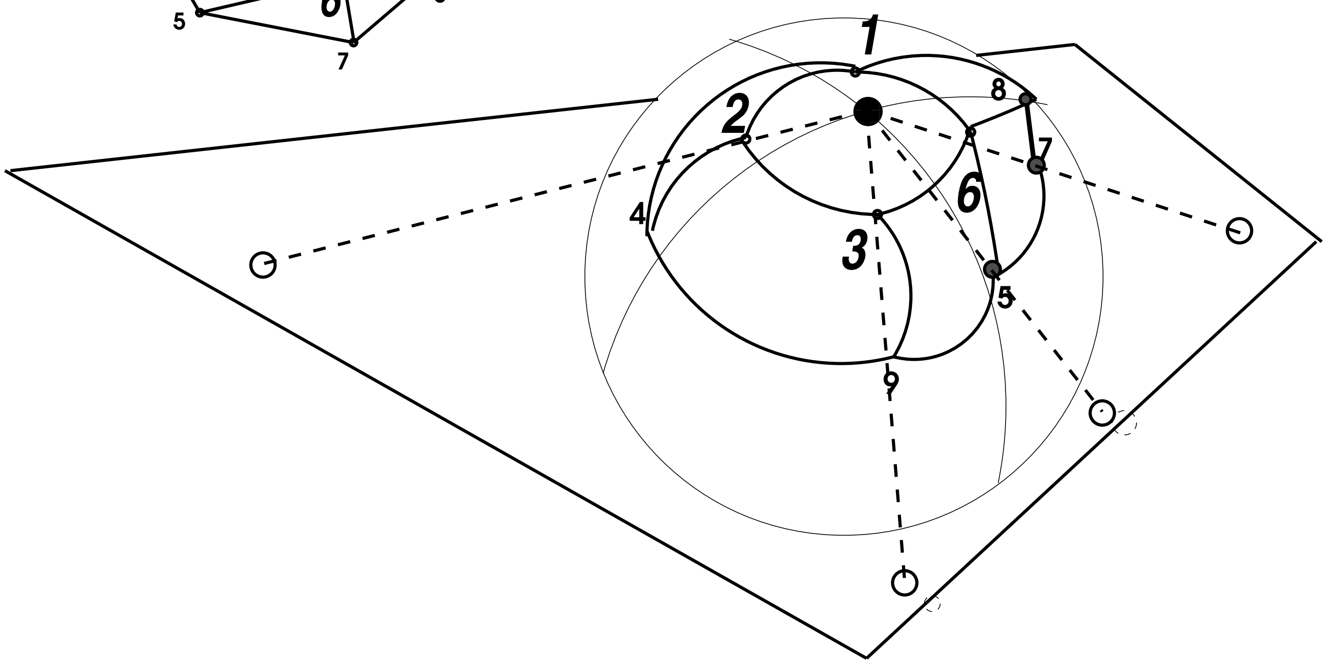
Proposition 1

In a plane graph, every cycle is the symmetric difference of the boundaries of some faces.

Proposition 2

For every face of a given plane graph G, there is a drawing of G for which the face is exterior.





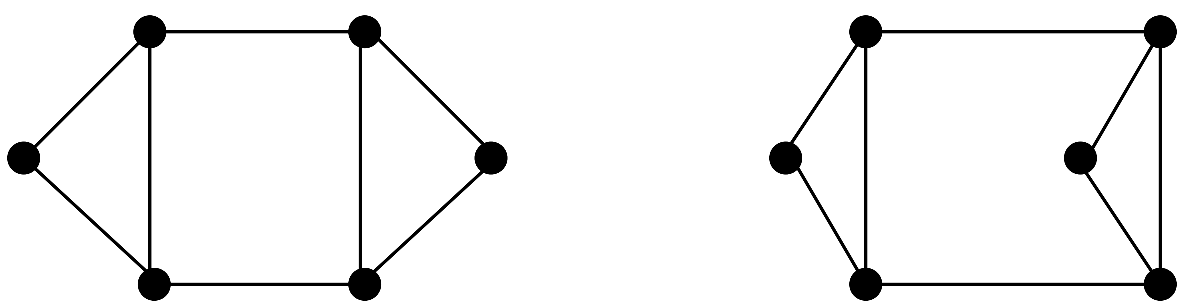
Dual Plane Graphs

Definition 5

Let G be a plane graph. The **dual graph** G^* of G is a new plane graph having a vertex for each face in G and the edges that correspond to the edges of G in the following manner:

> if e is an edge of G which separates two faces X and Y, then the corresponding dual edge $e^* \in E(G^*)$ is an edge joining the vertices x and y that correspond to X and Y respectively.

Remark. Different plane drawings (embeddings) of the same planar graph may have non-isomorphic duals.



Construct duals to these drawings.

Definition 6

The **length** of a face F in a plane graph G is the total number of edges in the closed walks in G that bound the face.

Proposition 3

If $l(F_i)$ denotes the length (*the number of edges in its boundary*), of face F_i in a plane graph G, then

$$2e(G) = \sum l(F_i).$$

Definition 7

A bond of a graph G is a **minimal non-empty edge cut.**

Proposition 4

Edges of a plane graph G form a cycle iff the corresponding edges in G^* form a bond.

Theorem 1 *(Euler):*

If G is a connected plane (p, q)-graph with r faces, then

$$p - q + r = 2.$$

Proof. We prove it by induction on q.

Base. If $q = 0$, then $p = 1$; obviously, $r = 1$, and the result follows.

Inductive Step. Assume that the Euler Theorem holds true for all connected graphs with fewer than q ($q \geq 1$) edges, and let G be a connected plane graph with q edges.

If G is a tree, then $p = q + 1$ and $r = 1$ (the only face is exterior) implying the result.

If G is not a tree, then it has an enclosed face. The edges of the face form a cycle. Take any edge e on the cycle and consider graph

$$H = G - e.$$

Since $q(H) = q - 1$, by induction, $p(H) - q(H) + r(H) = 2$. But $p(H) = p$ and $r(H) = r - 1$. The result follows. ∎

Theorem 2

If G is a planar graph (no parallel edges) with p vertices and q edges, $(q \geq 3)$, then $q \leq 3p - 6$. If, in addition, G is bipartite, then $q \leq 2p - 4$.

Proof. Let r be the number of faces of G and let m_i be the number of edges in the boundary of the i^{th} face $(i = 1, ..., r)$.

Since every face contains at least three edges,

$$3r \leq \sum_{i=1}^{r} m_i.$$

On the other hand, since every edge can be in the boundary of at most two faces,

$$\sum_{i=1}^{r} m_i \leq 2q.$$

Thus, $3r \leq 2q$ and by Euler's Theorem, $p - q + 2q/3 \geq 2$, implying $q \leq 3p - 6$.

If G is bipartite, the shortest cycle is of length at least 4. Thus,

$$4r \leq \sum_{i=1}^{r} m_i.$$

Together with $\Sigma_{i=1}^{r} m_i \leq 2q$ and $p - q + r = 2$, we get the second part of the Theorem.

Corollary 1 *Every planar graph G contains a vertex of degree at most 5.*

Proof. (HINT: assuming that all degrees are ≥ 6, estimate the number of edges in the graph, and compare your estimate with that by Theorem 2 .)

Corollary 2 *Graphs $K_{3,3}$ and K_5 are not planar.*

Proof. For K_5, $p = 5$ and $q = 10$. Thus, $q > 3p - 6 = 9$ and by Theorem 2, K_5 is not planar.

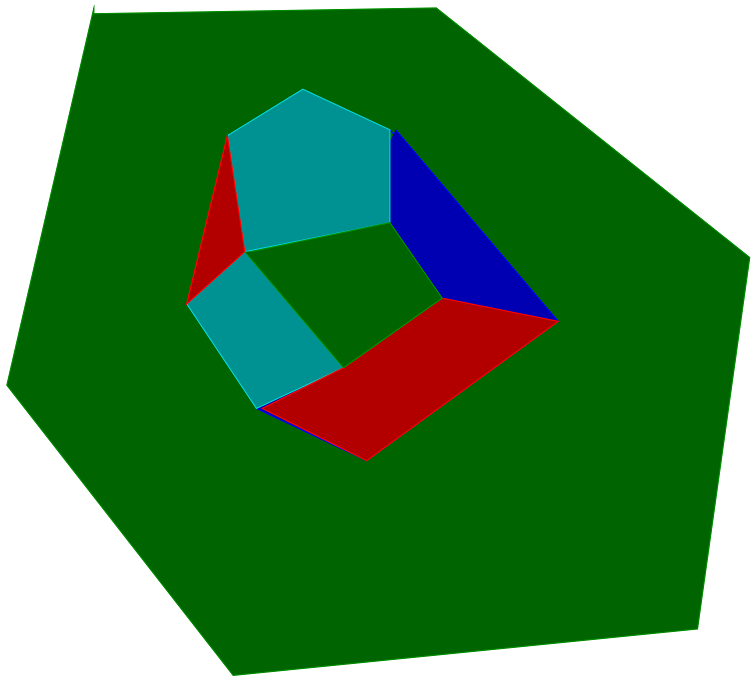
If $K_{3,3}$ were planar, then the second part of Theorem 2 would apply, leading to a contradiction, since for this graph $p = 6$, $q = 9$, and $q > 2p - 4$.

Definition 8 *A subdivision of an edge ab in a graph G is an operation which replaces ab with two edges az and zb where z is a new vertex different from other vertices of G. The result of the subdivision is also called a subdivision of G.*

A Kuratowski graph is a graph obtained by several subdivisions from either K_5 or $K_{3,3}$.

Corollary 3 *No Kuratowski graph is planar.*

2 Coloring Planar Graphs



Theorem 3 *Every planar graph G is 5-colorable.*

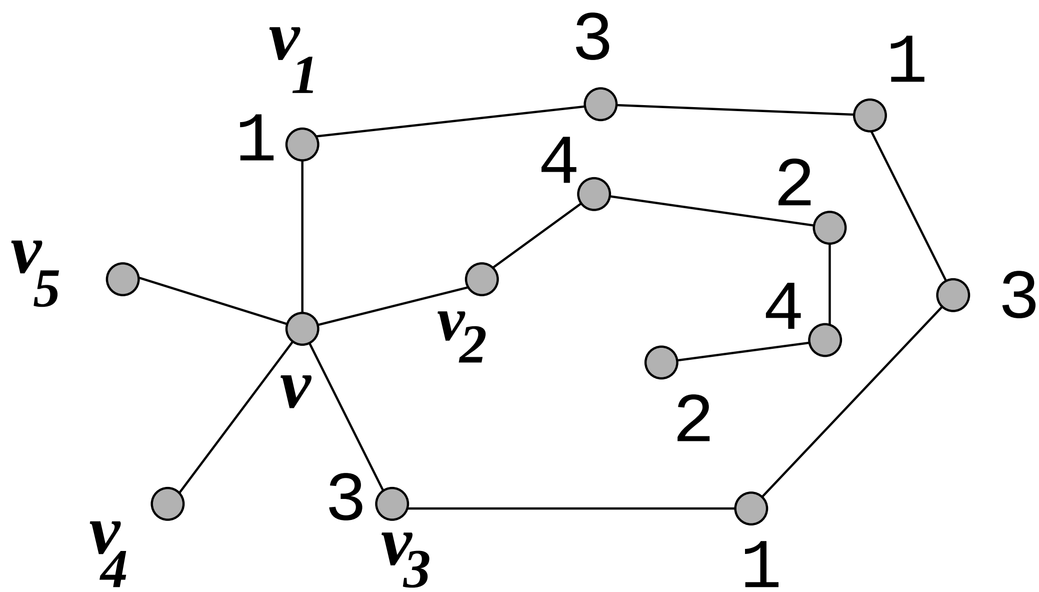
Proof. By induction on the number $n(G)$ of vertices.

Base. For all planar graphs with $n(G) \leq 5$, the statement is correct.

Inductive step. Let G have more than 5 vertices. Select a vertex v of degree ≤ 5. It always exists, since else, the number of edges in the graph would exceed the upper bound of $3p - 6$. By induction, graph $G - v$ is 5-colorable.

Consider a 5-coloring of $G - v$. If any color, 1 2, 3, 4, 5 is not used for vertices adjacent to v, use it for v. Thus, we need to assume that v has 5 neighbors that are colored using all 5 colors. Let us call those neighbors according to their colors: v_1, v_2, v_3, v_4, v_5 (see Figure below).

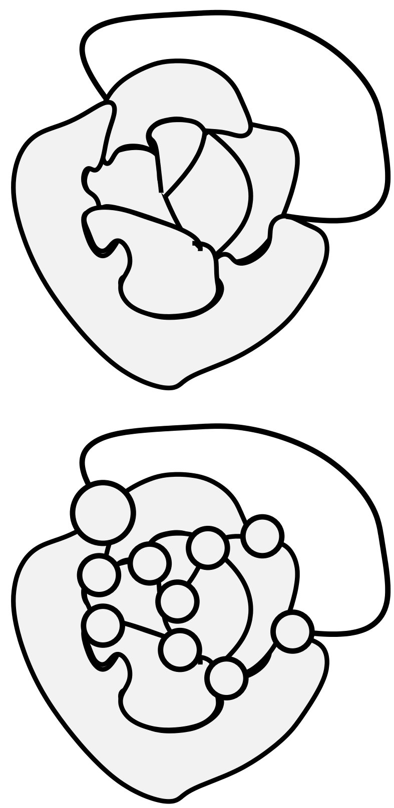
Consider a bipartite graph H induced by all vertices of $G - v$ whose colors are 1 or 3, and let C be the connected component of H which contains vertex v_1. If C does not contain vertex v_3, then we re-color C: every vertex of C whose color is 1 (resp. 3) gets color 3 (resp. 1). This recoloring frees color 1 for v, yielding a 5-coloring of G.



Finally, assume that C contains vertex v_3. Thus, there is a 2-colored path connecting vertices 1 and 3. This path together with vertex v forms a cycle which makes it impossible the existence of a path colored 2 and 4 connecting vertices v_2 and v_4. Thus, recoloring the 2-colored connected component containing vertex v_2 makes color 2 available for coloring v. ∎

From colorings faces of a map to coloring its edges.

The coloring of the faces of an arbitrary map is reduced to that of a map for which every vertex has three incident edges. See the Figure below.

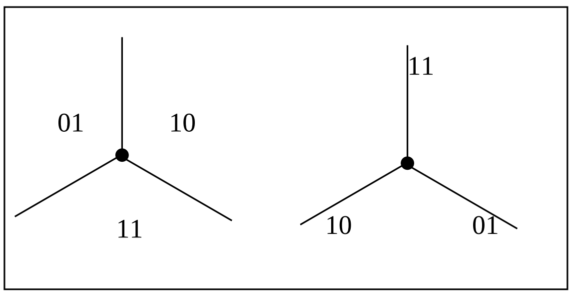


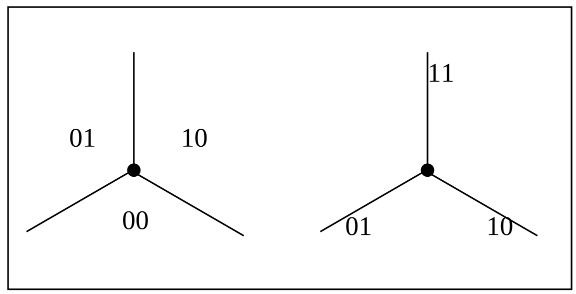
An edge coloring of a graph G is an assignment of colors to edges such that any two edges incident to the same vertex have different colors.

Theorem. (Tait, 1878). A 3-regular planar graph without bridges is 4-face colorable iff it is 3-edge-colorable.

Proof. Suppose G satisfies the conditions of the theorem and it is 4-face-colorable. Let the colors be $c_0 = 00,\ c_1 = 01,\ c_2 = 10,\ c_3 = 11.$

Because G is bridge-less, every edge bounds two distinct faces. Given an edge between faces colored c_i and c_j, assign this edge the color obtained by adding c_i and c_j coordinate-wise modulo 2. For example: $c_0 + c_1 = c_1;\quad c_1 + c_3 = c_2;\quad c_2 + c_3 = c_1.$





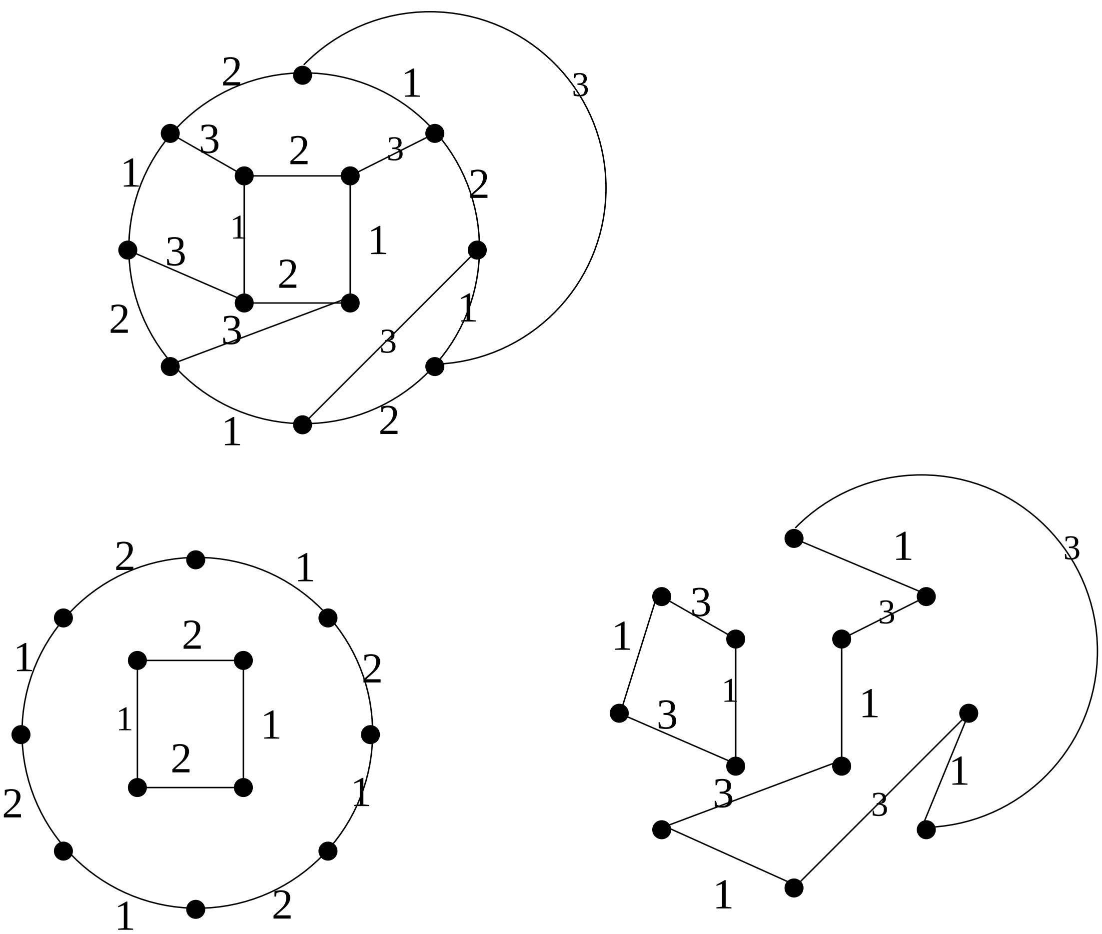
The reverse. Suppose now that G has a proper 3-edge-coloring, and let the colors be $1, 2$, and 3. Since G is 3-regular, every color appears at every vertex. Let E_1, E_2, and E_3 be the edge-sets colored $1, 2$, and 3, respectively. The union of any two of them is the union of disjoint cycles. Let $H_1 = E_1 \cup E_2$ and $H_2 = E_1 \cup E_3$.

The faces of graph H_1 can be 2-colored with two colors α and β.

The faces of graph H_2 can be 2-colored with two colors γ and δ.

Then each face of G is a subset of a face in H_1 and a subset of a face in H_2. Assign to each face in G a pair of colors (x, y) where $x \in \{\alpha, \beta\}$ and $y \in \{\gamma, \delta\}$.

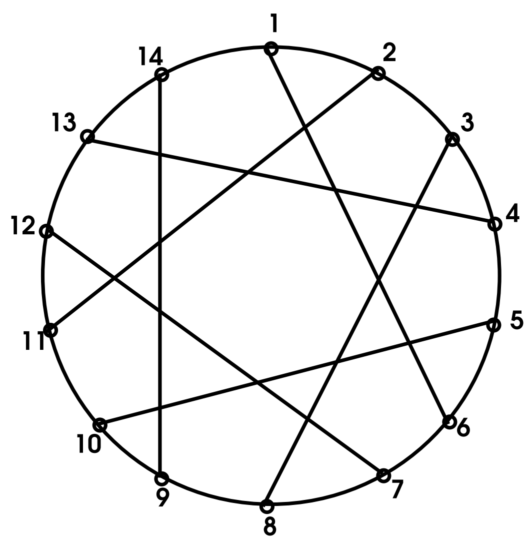
It is easy to prove that any two adjacent faces of G get pairs off colors that are differ in at least one coordinate. Thus the assignment is a 4-coloring of the faces of G.



Tutte's conjecture: every bridge-less 3-regular graph G which is not three-edge colorable has the Petersen graph as its minor, that is the Petersen graph can be obtained from G by contracting and removing edges.

Problem 1 *Use the fact that every cycle of the Petersen graph is of length 5 or more to prove that it is non-planar.*

Problem 2 *Show that the Heawood graph below is non-planar.*



Heawood graph

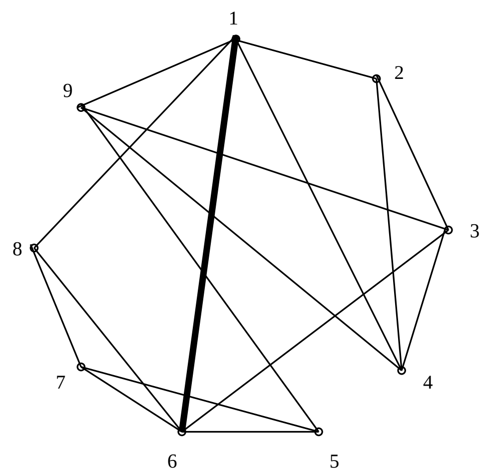
Problem 3 *Prove that a plane graph G is bipartite iff its dual G^* is Eulerian.*

Definition 9

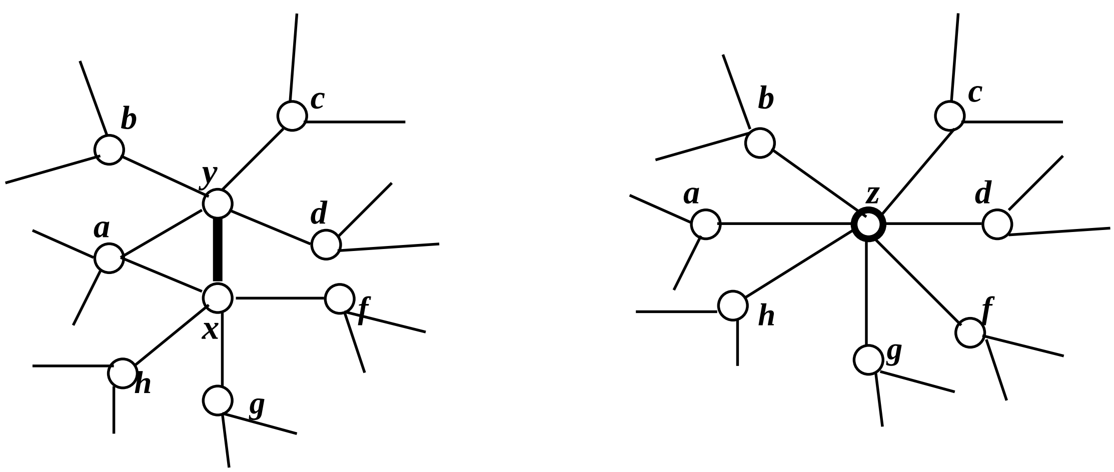
A *subdivision of an edge* $e = (xy)$ is an operation which removes e from the graph, adds a new vertex z along with two new edges: (xz) and (zy). A *subdivision of a graph* G is a graph that can be obtained from G by a sequence of edge subdivisions.

Definition 10

Two graphs G and H are called *homeomorphic* if either $G \approx H$, or there is a graph F such that both G and H can be obtained from F by two separate subdivisions of F.



Definition 11 *The operation of* **contraction** *of an edge $e = xy$ of a graph G removes from the graph e, x, and y, and adds a new vertex z, which is adjacent to any old vertex u iff u is adjacent to at least one of x or y. The resulting graph is denoted $G/(x, y)$.*



Problem 4 *If G is non-planar, then every subdivision of G is non-planar.*

Problem 5 *If G is planar, then every contraction of G is planar.*

Problem 6 *If the contraction $G/(x,y)$ has a subgraph homeomorphic to K_5 or $K_{3,3}$, then G has a subgraph homeomorphic to either K_5 or $K_{3,3}$.*

Definition 12 *A* **maximal** *planar graph is a simple planar graph which is not a spanning subgraph of another planar graph. A* **triangulation** *is a simple plane graph where every face is a 3-cycle.*

Problem 7 *For a simple plane graph G with n vertices, the following are equivalent*

1. *G has $3n-6$ edges;*
2. *G is a triangulation; and*
3. *G is a maximal planar graph.*

Problem 8 *Give two examples of planar graphs with no vertex of degree less than five.*

Problem 9 *Show that every planar graph of order ≥ 4 has at least four vertices of degree less than or equal to 5.*

Problem 10 *Show that* $K_5 - e$ *is planar for any edge* $e \in E(K_5)$. *Show that* $K_{3,3} - e$ *is planar for any edge* $e \in E(K_{3,3})$.

Problem 11 *A graph is called outerplanar, if it can be embedded into a plane so that every vertex of the graph lies on the boundary of then exterior face. Show that a planar* G *is outerplanar if and only if* $G + K_1$ *is planar.*