1 Matchings in Graphs
Definition 1

Two edges are called **independent** if they are not adjacent in the graph. A set of mutually independent edges is called a **matching**.

A matching is called

- **maximal** if no other matching contains it.
- **maximum** if its cardinality is maximal among all matchings
- **perfect** if every vertex of the graph is incident to an edge of the matching.

*maximal vs maximum*

![Diagram of maximal vs maximum matchings](image)

**How to construct a maximal matching?**
Given a matching $M$ in $G$,

a path is called $M$-alternating if its edges are alternatively in $M$ and not in $M$.

a vertex $x$ is called weak if no edge of $M$ is incident to $x$.

Using alternating path connecting weak vertices
The symmetric difference of two sets \( X \) and \( Y \) is the set of all elements that belong to one but not the other of the sets.

\[
X \otimes Y = (X \cup Y) - (X \cap Y)
\]

**Theorem 1** A matching \( M \) is maximum iff there exists no alternating path between any two distinct weak vertices of \( G \).

**Proof.** For a proof, we will try to answer the following questions:

- If a graph \( M \) is a matching, what is the maximum degree of a vertex in such a graph?

- If the edge set \( E \) of a graph \( F \) is the symmetric difference of two matchings \( M_1 \) and \( M_2 \), then what is the maximal vertex degree of \( F \)?

- Consider a component \( C \) of \( F \) above. Can \( C \) be a path, a cycle, anything else?

- If \( C \) is a cycle of \( F \), can it have an odd length; an even length?

**Conclusion:** Let \( M \) be a matching which is not maximum and let \( M^* \) be maximum. Then at least one connected component of \( M \otimes M^* \) is an alternating path containing more edges from \( M^* \).

That component is an augmenting path for \( M \).  \( \blacksquare \)
Definition 2

An edge cover of graph $G$ is a set $L$ of edges such that every vertex is an endpoint of an edge in $L$.

$\beta'(G)$ is the smallest size of an edge cover.

$\alpha'(G)$ is the largest size of a matching in $G$.

Theorem 2

For every graph $G$ without isolated vertices,

$$\alpha'(G) + \beta'(G) = n(G).$$

Proof.

$\beta'(G) \leq n(G) - \alpha'(G)$. Starting with a matching, add one edge for every unsaturated vertex. The total is $\alpha'(G) + n - 2\alpha'(G) = n - \alpha'(G)$, a cover size.

\[
\begin{align*}
\text{maximum matching} & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{unsaturated vertices} &
\end{align*}
\]

$\alpha'(G) \geq n(G) - \beta'(G)$. Consider the minimal edge cover $L$ and the subgraph $H = (V, L)$.

$H$ doesn’t have paths of length $> 2$, since otherwise an edge inside could be removed yielding a smaller edge cover. Thus, every connected component of $H$ is a star.

If $k$ is the number of these stars, $|L| = n(G) - k$. Form a matching by picking one edge from each star.
2 Matchings in Bipartite Graphs

Definition 3
A graph $G(V,E)$ is called **bipartite** if $V$ can be partitioned into two subsets $V_1$ and $V_2$ so that $E \subseteq V_1 \times V_2$.

**Proposition 1** A graph $G$ is bipartite if and only if it has no cycles of odd length.

**Proof.** If $G$ is bipartite and $C$ is a cycle of $G$ then the vertices of $C$ can be labeled by 0 and 1 depending to which part the vertex belongs to. This implies that the length of the cycle is even.

Let now $G$ be a graph without cycles of odd length. We want to prove that $G$ is bipartite. Obviously, we may assume that $G$ is connected. Then, consider an arbitrary vertex $v \in V(G)$ and define $V_{even}$ (resp. $V_{odd}$) to be the set of all vertices whose distance from $v$ is even (resp. odd). If there were an edge connecting two vertices from $A_{even}$ (resp. two vertices from $A_{odd}$), then $G$ would have an odd cycle. Thus, $G$ is bipartite and $(V_{even}, V_{odd})$ is the partition. 

Definition 4
Given a matching $M$ of a bipartite graph $G = (V_1, V_2; E)$, $V_1(M)$ (resp. $V_2(M)$) denotes the set of vertices in $V_1$ (resp. $V_2$) incident to the edges in $M$. 


Definition 5

A matching \( M \) is said to **saturate** a vertex \( v \), if there is an edge \((v, w) \in M\).

**Theorem 3** A bipartite graph \( G = (V_1, V_2, E) \) contains a matching with \(|V_1|\) edges (a matching saturating all \( V_1 \)) iff

\[
\forall X \subseteq V_1, \ |X| \leq |E(X)|. \tag{*}
\]

**Proof.** First, we prove that the existence of a matching saturating all \( V_1 \) implies \(|X| \leq |E(X)|\).

Indeed, since \( M \) is a matching, the sets \( V_1(M) \) and \( V_2(M) \) have the same cardinality. For every \( x \in X \) there is a distinct vertex \( y \in E(X) \) for which \((x, y)\) is in \( M \).

Now we prove that condition

\[
\forall X \subseteq V_1, \ |X| \leq |E(X)| \tag{*}
\]

implies the existence of a matching which saturates all \( V_1 \).

Applying (*) to \( X = V_1 \), we get \(|V_1| \leq |V_2|\). Our goal is to prove that there is a matching which saturates all vertices in \( V_1 \).

Suppose \( M_0 \) is a maximum matching and it doesn’t saturates all \( V_1 \). We will show that this supposition leads to a contradiction to the condition (*). The idea is illustrated in the next figure.
Let $S = V_1 - V_1(M_0)$ and $T = V_2 - V_2(M_0)$. If there were an edge $(x, y) \in E$ such that $x \in S$ and $y \in T$, the new matching is $M_0 \cup \{(x, y)\}$ contrary to the maximality of $M_0$.

So, we assume that no vertex in $S$ is adjacent to any vertex in $T$.

Because of (*), for every vertex $v \in S$, there are edges incident to $v$. Consider any path in $G$ satisfying the following conditions:

1. the first vertex of the path is in $S$;
2. the edges of the path alternate between edges not in $M_0$ and edges in $M_0$.

**Claim.** No path satisfying (1) and (2) contains a vertex in $T$.

Indeed, if such a path had a vertex in $T$, it would be its last vertex, and the path itself would be an $M_0$-alternating path with two end-vertices that are not saturated by $M_0$. Thus, the path would have been augmenting, yielding a matching $M_1$ which is larger than $M_0$ contrary to the maximality of $M_0$. 


Let $v \in S$ and $D$ be the set of all vertices reachable from $v$ by $M_0$-alternating paths.

Because of the Claim, $D \cap T = \emptyset$. Furthermore,

$$|V_1(M_0) \cap D| = |V_2(M_0) \cap D|.$$

Thus, for $X = |V_1(M_0) \cap D| \cup \{v\}$,

$$|E(X)| = |V_2(M_0) \cap D|,$$

which implies that

$$|E(X)| = |X| - 1.$$

This contradicts the condition on $G$ and thus proves that $S = \emptyset$.  \hfill \Box
Definition 6

Given a graph $G(V, E)$, a subset $C \subseteq V$ is called vertex cover, if every edge in $E$ is incident to at least one vertex in $C$.

**Question:** if $C$ is a vertex cover for $G$, what kind of graph will be obtained if $C$ is removed from $G$?

**Theorem 4** (Egervári (1931); König (1931)) A maximum cardinality of a matching in a bipartite graph $G = (V_1, V_2, E)$ is equal to the minimum cardinality of a vertex cover.

**Proof.**

$\Rightarrow$ Let $c$ be the smallest size of a vertex cover and $m$ be the largest size of a matching in $G$. Then $c \geq m$, since in any cover, every edge in a matching must be “covered” by its own vertex.

$\Leftarrow$ Given a minimum vertex cover $C$, construct a matching of size $|C|$, which proves that $m \geq c$. 


Split the graph into two subgraphs using the minimum vertex cover

The left subgraph (resp. right) satisfies the condition
\[ |X| \leq |E(X)| \]

The left subgraph (resp. right) must have a matching which saturates partition \( C \cup V_1 \) (resp. \( C \cup V_2 \))
Constructing maximum matchings in bipartite graphs

The algorithms is the iteration of a procedure, **AUGMENT** which starts with a matching (can be empty) of a bipartite graph. **AUGMENT** either outputs a bigger matching, or a vertex cover of the size of the matching, which proves that the current matching is maximum in the graph. The last iteration ends with the construction of the cover set.

**Procedure AUGMENT**

**Input:** graph $G(V_1, V_2; E)$ and a matching $M$.

let $M(V_1)$ (resp. $M(V_2)$) denote the vertices in $V_1$ (resp. $V_2$) that are saturated by $M$.

if $M(V_1) = V_1$,
then halt with $M$ as a maximum matching.
and $V_1$ as a minimum vertex cover.

else
let $U = V_1 - M(V_1)$;
construct an alternating tree $\mathcal{T}(U)$;
if $\mathcal{T}(U)$ contains vertices in $V_2 - M(V_2)$,
extract an augmenting path $P(v_1, v_2)$
($v_1 \in V_1 - M(V_1)$; $v_2 \in V_2 - M(V_2)$);
augment $M$;
else
let $S$ be the set of all vertices in $V_1$ that are reached by $\mathcal{T}$;
let $T$ be the set of all vertices in $V_2$ that are reached by $\mathcal{T}$;
halt with maximum matching $M$ and minimum vertex cover $(V_1 - S) \cup T$. 
Illustration of the algorithm for constructing a maximum matching in a bipartite graph.