1 Coloring Geographic Maps

The four color map theorem states that

given any separation of the plane into contiguous regions, such as a political map of the states of a country, the regions can be colored using at most four colors so that no two adjacent regions have the same color. Two regions are called adjacent only if they share a border segment, not just a point.

The conjecture was first proposed in 1852 when Francis Guthrie, while trying to color the map of counties of England, noticed that only four different colors were needed.
The four color theorem was the first major theorem to be proven using a computer. In 1976 Kenneth Appel and Wolfgang Haken showed a particular set of 1,936 maps that have two properties:

(a) at least one must be included in any map (the set is unavoidable) and
(b) each one cannot be part of the smallest counterexample (each is reducible).

Thus, there is no smallest counterexample, and hence no counterexample at all. Appel and Haken used a special-purpose computer program, and hundreds of pages of hand analysis, to show their group of maps had these two properties.

A simpler proof using the same ideas and still relying on computers was published in 1997 by Robertson, Sanders, Seymour, and Thomas.

Additionally in 2005, the theorem was proven by Georges Gonthier with general purpose theorem proving software.
Definition 1

A $k$-vertex coloring, or simply a $k$-coloring of a graph $G$ is a mapping

$$f : V(G) \rightarrow \{\text{set of } k \text{ elements called colors}\},$$

such that adjacent vertices are mapped onto different elements. Often, the colors are integers $1, 2, \ldots, k$.

A graph is called $k$-colorable, if it has a $k$-coloring. The chromatic number of a graph $G$, $\chi(G)$, is the minimal $k$ for which the graph is $k$-colorable. A graph with chromatic number $k$ is called $k$-chromatic.

Let $\chi(G) = k$. $G$ is called $k$-color-critical, or $k$-critical, if for every proper subgraph $H \subset G$,

$$\chi(H) < k.$$

Find the chromatic numbers of the graphs below.
Proposition 1
A $k$-coloring of $G(V, E)$ is a partitioning

$$V = V_1 \cup V_2 \cup \cdots \cup V_k$$

into at most $k$ independent sets of $G$.

**Theorem 1** The vertex coloring problem is NP-hard.

Lower bounds for the chromatic number of a graph.

**Proposition 2**

If $\alpha(G)$ is the largest size of an independent set in $G$, then

$$\chi(G) \geq \frac{|V|}{\alpha(G)}.$$

**Proposition 3**

If $\omega(G)$ is the size of the largest clique in $G$, then

$$\chi(G) \geq \omega(G).$$
3 Bounds for the chromatic number

A $k$-coloring of a graph $G(V, E)$ is a partitioning of $V$ into $k$ independent set of $G$: $V = I_1 \cup I_2 \cup \ldots \cup I_k$. \( \Rightarrow \chi(G) \geq |V|/\alpha(G) \).

\[\begin{align*}
\alpha(G) &= p \\
\omega(G) &= p
\end{align*}\]

An example: Mycielski’s graphs.

\[\begin{align*}
M_1 & \quad \alpha(G) = 2; \quad \omega(G) = 2; \quad \chi(G) = 3; \\
M_2 & \quad \alpha(G) = 5; \quad \omega(G) = 2; \quad \chi(G) = 4;
\end{align*}\]

\textbf{Theorem 2} The chromatic number $\chi(M_k) = k + 2$. 
Upper bounds for the chromatic number of a graph.

**Theorem 3**

For every graph $G$, $\chi(G) \leq \Delta(G) + 1$.

**Proof.** The *greedy* algorithm for coloring in $\Delta(G) + 1$ colors.

\[
\text{for } i = 1, \ldots, n \\
\text{color vertex } v_i \text{ using the smallest available positive integer } i \leq \Delta(G) + 1
\]

The algorithm and the bound are correct because

- every vertex has at most $\Delta(G)$ neighbors $\implies$
- every vertex has at most $\Delta(G)$ neighbors that are colored $\implies$
- there is always at least one color available

**Question:** Are there graphs for which $\chi(G) = \Delta(G) + 1$?

**Theorem 4** (Brooks [1941]) *For every graph $G$ which is not complete and not a cycle of an odd length, $\chi(G) \leq \Delta(G)$.***
Proof (Lovász [1973]) We can assume that \( \Delta(G) \geq 3 \).

Let \( G \) be a minimal counterexample to the statement. Then

**Claim 1.** \( G \) is connected. Obvious.

**Claim 2.** \( G \) is 2-connected.

**Proof.** If \( x \) is a cut vertex, and \( C_1, C_2, \ldots, C_k \) are the connected components of \( G - x \), then by the minimality of \( G \), each \( C_i \cup \{x\} \) \((i = 1, \ldots, k)\) can be \( \Delta \)-colored (none is a complete graph on \( \Delta + 1 \) vertices). By renumbering the colors (if needed) we get all \( \{C_i \cup \{x\}\} \) colored so that \( x \) has the same color for all of them. (see the Figure below). This yield a \( \Delta \)-coloring of \( G \).  

**Claim 3.** \( G \) is \( \Delta \)-regular.

**Proof.** If \( G \) is not regular, let us reorder the vertices \( x_1, x_2, \ldots, x_n \) so that
(a) $x_n$ is the vertex of the smallest degree $< \Delta$; and
(b) for each $i < n$, there is an edge $x_ix_j$ where $j > i$.
It is possible since $G$ is connected.

Apply the greedy coloring to $G$ according to the new order of the vertices. Because of property (b), every vertex $x_i$, where $i < n$, gets a color $\leq \Delta$; because of property (a), the color of $v_n$ is also $\leq \Delta$.

Claim 4. For any $v \in V(G)$, there are two non-adjacent neighbors of $v$.
Proof. Else $G$ is $K_{\Delta+1}$. 

Claim 5. If there is a vertex $v$ with two non-adjacent neighbors $x$ and $y$ such that $G - \{x, y\}$ is connected, then $G$ is $\Delta$-colorable.
Proof. Reorder the vertices so that
(a) \( x_n = v; \)
(b) \( x_1 = x, x_2 = y, \) and for each \( i \in [3, n - 1], \) there is an edge \( x_ix_j \) where \( j > i. \) Such an order is possible since \( G - \{x, y\} \) is connected.

Apply the greedy coloring to \( G \) according to the new order of the vertices.

Since \( x_1 \) and \( x_2 \) are not adjacent, they get the same color 1. Every vertex \( x_i \) \((i \in [3, n - 1])\) gets a color \( \leq \Delta \) because of the property \( b. \) The color of \( x_n = v \) is also \( \leq \Delta \) since two of its neighbors have the same color.

**Claim 6.** A triple \( \{v, x, y\} \) exists such that \( G - \{x, y\} \) is connected; and \( x, y \) are non-adjacent neighbors of \( v. \)

Consider any vertex \( u \) and two of its non-adjacent neighbors \( u_1, u_2. \)

If \( G - \{u_1\} \) is 2-connected then let \( v = u, x = u_1 \) and \( y = u_2. \)

Let \( G - \{u_1\} \) be 1-connected. Since \( G \) is 2-connected, \( u_1 \) is adjacent to at least two distinct, not cut-vertices of leaf-blocks of \( G - \{u_1\}. \) Then \( v = u_1 \) and two neighbors of \( u_1 \) in two leaf-blocks are set to be \( x \) and \( y. \)
4 Several problems.

**Problem 1** If \( \{G_i\}_{i=1}^k \) are the blocks of graph \( G \), then \( \chi(G) = \max_i \chi(G_i) \).

**Problem 2** The **Cartesian product** of two graphs \( G \) and \( H \), denoted \( G \otimes H \), is the graph whose vertex set is \( V(G) \times V(H) \) and the edge set \( E(G \otimes H) \) consists of all pairs \((u,v)(u',v')\) such that either \( u = u' \) & \( vv' \in E(H) \) or \( uu' \in E(G) \) & \( v = v' \).

Draw the graph \( K_{1,3} \otimes P_3 \) and exhibit an optimal coloring of it. Draw \( C_5 \otimes C_5 \) and find a proper 3-coloring of it with color classes of sizes 9, 8, and 8.

**Problem 3** Prove or disprove: every \( k \)-chromatic graph \( G \) has a proper \( k \)-coloring in which some color class has \( \alpha(G) \) vertices.

**Problem 4** Prove or disprove: for every connected graph \( G \), \( \chi(G) \leq 1 + av(G) \), where \( av(G) \) is the average vertex degree in \( G \).

**Problem 5** Given a graph \( G \), a \( G \)-subdivision is obtained from \( G \) by successive edge subdivisions (a sub-
division of an edge $xy$ is a replacement of the edge with a path $xe_1ze_2y$).

Find a subdivision of $K_4$ in graph $M_2$.

**Problem 6** Let $\phi$ be a $k$-coloring of a $k$-chromatic graph $G$. Prove that for every color $i \in [1, k]$, there is a vertex colored $i$ which is adjacent to vertices of all other $k - 1$ colors.

**Problem 7** Let $G$ be a $k$-critical graph. Prove that for no two vertices $x, y \in V(G)$, $N(x) \subseteq N(y)$. Conclude that every $k$-critical graph which is not complete, contains $> k + 1$ vertices.

**Conjecture 1** (Hadwiger, 1943) If $G$ is a connected graph with $\chi(G) = n$, then $G$ contains $n$ disjoint connected subgraphs such that each subgraph is connected by an edge to each other subgraph.

**Conjecture 2** (Erdős-Faber-Lovász; 1980) The union of $n$ pairwise edge-disjoint copies of $K_n$ is $n$-colorable.
Conjecture 3 (Reed, 1998) For any graph $G$, 
\[
\chi(G) \leq \left\lceil \frac{\Delta(G) + \omega(G) + 1}{2} \right\rceil.
\]