Clusters in a multigraph with elevated density*

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Abstract

In this paper, we prove that in a multigraph whose density Γ exceeds the maximum vertex degree Δ, the collection of minimal clusters (maximally dense sets of vertices) is cycle-free. We also prove that for multigraphs with $\Gamma > \Delta + 1$, the size of any cluster is bounded from the above by a function, which depends on Δ and Γ only. Finally, we show that two well-known lower bounds for the chromatic index of a multigraph are identical.

1 Introduction

The **chromatic index** $\chi'(G)$ of a multigraph $G(V, E)$ is the minimal number of colors needed to color all edges of G so that no two edges incident to the same vertex have the same color. A trivial lower bound for $\chi'(G)$ is

$$\Delta(G) \leq \chi'(G),$$

where $\Delta(G)$ is the maximal vertex degree in G. A remarkable result discovered by Vizing (see [11]) gives the upper bound $\chi' = \chi'(G) \leq \Delta(G) + p(G)$, where $p(G)$ is the maximal number of parallel edges in G. Thus, for multigraphs without parallel edges (graphs), there are just two possible values for χ': either Δ, or $\Delta + 1$.

For general multigraphs, $p(G) \geq 1$, Shannon proved in [8] that $\chi'(G) \leq \lfloor (3\Delta)/2 \rfloor$, which is strengthened by Vizing's bound to $\chi' \leq \Delta + \min\{p, \lfloor \Delta/2 \rfloor\}$. The basic question in the multigraph edge-coloring is: "*what properties of a multigraph cause its chromatic index* χ' *to exceed* Δ?" Although to determine if $\chi'(G) = \Delta(G)$, or $\chi'(G) \geq \Delta(G) + 1$ is NP-complete, as proved by Hoyler ([4]), it is suspected that multigraphs with $\chi'(G) > \Delta(G) + 1$ can be completely characterized in terms of its **density** $\Gamma(G)$, defined by

$$\Gamma(G) = \max_{H \subseteq G} \lceil \frac{e(H)}{\lfloor v(H)/2 \rfloor} \rceil,$$

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where H is a sub-multigraph of G and $v(H)$ (resp. $e(H)$) denotes the number of vertices (resp. edges) in H. It is easy to see that for every multigraph G, $\Gamma(G) \le \chi'(G)$. Currently, no multigraph is known with $\chi'(G) > \max(\Delta + 1, \Gamma)$. Conjectures connecting the chromatic index, maximal degree, and the density of a multigraph were independently proposed by Goldberg ([2]) and Seymour ([9]) more than 25 years ago (see also [5]).

Conjecture 1 (Goldberg ([2]) For every multigraph G, if $\chi'(G) > \Delta + 1$, then $\chi'(G) = \Gamma(G)$.

Conjecture 2 (Seymour [9]) For every multigraph G, $\chi(G) \le \max\{\Delta(G), \Gamma(G)\} + 1$.

An extension of the conjectures above was proposed by Goldberg in [3]:

Conjecture 3 If $\Delta \ne \Gamma$, then $\chi' = \max\{\Delta, \Gamma\}$, else $\chi' \le \Delta + 1$.

Since all three conjectures are close to each other, we refer to them as the GS-conjecture. See [1], [3], [6], [7], [10] for some results towards the conjecture; in particular, Nishizeki and Kashiwagi ([7]) proved $\chi' = \Gamma$ for multigraphs with $\chi' > (11\Delta - 8)/10$.

The GS-conjecture motivates the study of the multigraphs with $\Gamma > \Delta$; we call them multigraphs with the elevated density. The properties of the multigraphs with the elevated density presented here are formulated in terms of two new notions: set-cycles and multigraph clusters.

Definition 1 *A sequence $\mathcal{S} = \{S_i\}_{i=1}^k$ of sets is called a set-cycle if*

$$\forall i \in [1, k], S_i \cap S_{i+1} \ne \emptyset \ \& \ S_i \cap S_{i+1} \cap S_{i+2} = \emptyset.$$

Here $S_{k+1} = S_1$ and $S_{k+2} = S_2$.

A collection $\mathcal{T} = \{T_j\}_{j=1}^m$ is called a set-forest, if no sequence of sets from $\mathcal{T}$ is a set-cycle.

Definition 2 *Given a multigraph $G(V, E)$, a set $S \subseteq V$ is called maximally dense, or a cluster, if $e(S) > (\Gamma - 1)\lfloor |S|/2 \rfloor$. A cluster S is called minimal if no proper subset of S is a cluster.*

Our main result (Section 2) establishes that the collection of minimal clusters in a multigraph with $\Gamma > \Delta$ has a simple structure: it is a set-forest. We also prove that in a multigraph with $\Gamma > \Delta + 1$, the size of any cluster is bounded by a function which depends on Γ and Δ only (not on the number of the vertices of the multigraph). This bound matches the upper bound of the size of the critical multigraph which was proved in [2] under the assumption of the GS-conjecture.

A lower bound for $\chi'(G)$, which is sometimes stronger than $\Gamma(G)$ can be formulated in terms of maximum matchings of subgraphs of G.

Definition 3 *Let $F \subseteq E$ and let $m(F)$ denote the maximal size of a matching comprised of edges in F. Then,*

$$\Omega(G) = \max_{F \subseteq E(G)} \lceil \frac{|F|}{m(F)} \rceil.$$

It is easy to see that $\Gamma(G) \leq \Omega(G) \leq \chi'(G)$. A star is an example of a multigraph with $\Gamma(G) < \Omega(G)$. If there were multigraphs with $\Delta \leq \Gamma < \Omega$, the GS-conjecture would be disproved. However, in Section 3, we use Tutte's matching theorem to prove that for every multigraph G, $\Omega(G) = \max\{\Delta(G), \Gamma(G)\}$.

The following notations are used in this paper. Given a set $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced on S. If $F \subseteq E(G)$, then $G[F]$ denotes the subgraph of G induced on F: the vertex set of $G[F]$ is the set of vertices incident to the edges in F, and the set of edges of $G[F]$ is set F. The degree of a vertex x is denoted $deg(x)$; given $S, T \subseteq V(G)$, $deg(S,T)$ denotes the number of edges xy such that $x \in S$ and $y \in T$; $deg_S(x)$ denotes the degree of x in the subgraph $G[S]$ induced on S; $\delta_S(x) = deg(x) - deg_S(x)$; $\delta(S) = \sum_{x \in S} \delta_S(x) = deg(S, V(G) - S)$; $\nabla(x) = \Delta - deg(x)$; and $\nabla(S) = \sum_{x \in S} \nabla(x)$.
All notions and notations not explained but used in the paper, can be found in [12].

2 Topology of minimal clusters.

The goal of this section is to establish several structural properties of the set of minimal clusters in a multigraph G with $\Gamma = \Gamma(G) > \Delta(G) = \Delta$. We also give an upper bound for the size of any cluster in a multigraph with $\Gamma(G) > \Delta + 1$; it turns out, for such multigraphs, the cluster size is bounded from the above by a function depending on Δ and Γ only. Throughout this section, G is a multigraph with $\Gamma(G) > \Delta(G)$. The first lemma is a simple extension of the standard inequality $2e(S) \leq \Delta|S|$.

Lemma 1 *For every subset $S \subseteq V(G)$,*

$$\delta(S) + \nabla(S) + 2e(S) = \Delta|S|. \tag{1}$$

Proof. The result follows from

$$\begin{aligned} 2e(S) &= \sum_{x \in V(S)} deg_S(x) = \sum_{x \in V(S)} (deg(x) - \delta_S(x)) \\ &= \sum_{x \in V(S)} (\Delta - \nabla(x) - \delta_S(x)) \\ &= |V(S)|\Delta - \nabla(S) - \delta(S). \qquad \blacksquare \end{aligned}$$

Lemma 2 *The cardinality of every cluster S in G is odd.*

Proof. If $|S|$ were even, then the defining inequality $(\Gamma - 1)\lfloor |S|/2 \rfloor < e(S)$ could be rewritten as

$$(\Gamma - 1)|S| < 2e(S).$$

Since $2e(S) \le \Delta|S|$, it would imply

$$(\Gamma - 1)|S| < 2e(S) \le \Delta|S|,$$

which contradicts our assumption $\Delta < \Gamma$. ∎

Lemma 3 *For every cluster* S,

$$\delta(S) + \nabla(S) \le \Delta - 2 - (\Gamma - \Delta - 1)(|S| - 1). \tag{2}$$

Proof. Since $|S|$ is odd,

$$(\Gamma - 1)\lfloor \frac{|S|}{2} \rfloor = (\Gamma - 1)\frac{|S| - 1}{2} < e(S).$$

This implies

$$(\Gamma - 1)\frac{|S| - 1}{2} + 1 \le e(S) \text{ and } (\Gamma - 1)(|S| - 1) + 2 \le 2e(S).$$

Combining the latter with $\Delta|S| - \nabla(S) - \delta(S) = 2e(S)$ (by Lemma 1), we have

$$(\Gamma - 1)(|S| - 1) + 2 \le \Delta|S| - \nabla(S) - \delta(S),$$

which is equivalent to (2). ∎

Lemma 4 *For any two minimal clusters* S_1 *and* S_2 *of a multigraph* G*, if* $S_1 \cap S_2 \ne \emptyset$*, then* $|S_1 \cap S_2|$ *is odd.*

Proof. Let us assume that $|S_1 \cap S_2| = 2a$, where a is a positive integer. Denote $A = S_1 \cap S_2$; $e_o = e(G[A])$; $2p_i + 1 = |S_i|$; $e_i = e(G[S_i - A])$, and $w_i = deg(A, S_i - A)$ $(i = 1, 2)$. By definition of a cluster, $(\Gamma - 1)p_i < e_i + w_i + e_0$ $(i = 1, 2)$, implying

$$(\Gamma - 1)(p_1 + p_2) < e_1 + e_2 + w_1 + w_2 + 2e_0. \tag{3}$$

Since $|S_i - A| = 2p_i - 2a + 1$, by minimality of cluster S_i, $e_i \le (p_i - a)(\Gamma - 1)$ $(i = 1, 2)$, hence

$$e_1 + e_2 \le (p_1 + p_2 - 2a)(\Gamma - 1) = (p_1 + p_2)(\Gamma - 1) - 2a(\Gamma - 1). \tag{4}$$

By Lemma 1, $w_1 + w_2 + 2e_0 \le 2a\Delta \le 2a(\Gamma - 1)$. Plugging it into (4), we obtain

$$e_1 + e_2 + w_1 + w_2 + 2e_0 \le (\Gamma - 1)(p_1 + p_2),$$

which contradicts inequality (3). ∎

It is easy to construct examples of minimal clusters that intersect. The multigraph in Figure 1 shows that the intersection of two minimal clusters can have more than one vertex.

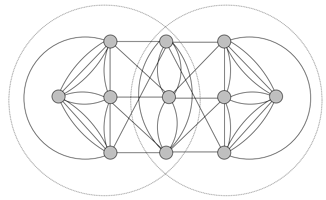

Figure 1: The intersection of two minimal clusters within dotted circles consists of three vertices; for the multigraph, $\Gamma = 9$; and $\Delta = 8$.

Theorem 1 *The set $\mathcal{T} = \{S_i\}_{i=1}^m$ of all minimal clusters in a multigraph G is a set-forest.*

Proof. Suppose that contrary to the statement, there is a set-cycle $\{S_i\}_{i=1}^k$ all of whose sets are minimal clusters in G. Denote $A_i = S_i \cap S_{i-1}$ and $B_i = S_i - A_i - A_{i+1}$ ($i \in [1,k]$). As before, we use indices "cyclically:" thus, $A_1 = S_1 \cap S_k$ and $B_k = S_k - A_k - A_1$.

Since $|S_i|$ and $|A_i|$ are odd (Lemmas 2 and 4), and $A_i \cap A_{i+1} = \emptyset$ (from the definition of a set-cycle), it follows that $|B_i| = |S_i| - |A_i| - |A_{i+1}|$ is also odd ($i \in [1,k]$).

Let $|A_i| = 2a_i + 1$, $|B_i| = 2b_i + 1$, $w_i^+ = deg(A_i, S_i - A_i)$, and $w_i^- = deg(A_i, S_{i-1} - A_i)$ ($i \in [1,k]$). Clearly,

$$e(S_i) \le e(A_i) + e(B_i) + e(A_{i+1}) + w_i^+ + w_{i+1}^-.$$

Since A_i and B_i are proper subsets of minimal clusters.

$$e(A_i) \le (\Gamma - 1)a_i \text{ and } e(B_i) \le (\Gamma - 1)b_i \ (i \in [1,k]).$$

Thus,

$$\begin{aligned} \sum_{i=1}^k e(S_i) &\le \sum_{i=1}^k ((\Gamma-1)a_i + (\Gamma-1)b_i + (\Gamma-1)a_{i+1}) + \sum_{i=1}^k (w_i^+ + w_{i+1}^-) \\ &= (\Gamma-1)\sum_{i=1}^k (2a_i + b_i) + \sum_{i=1}^k (w_i^- + w_{i+1}^+). \end{aligned}$$

Since, $w_i^- + w_{i+1}^+ \le \delta(S_i)$ and, by Lemma 2, $\delta(S_i) \le \Delta - 2 \le \Gamma - 1$, after simplifications we have

$$\sum_{i=1}^k e(S_i) \quad \le \quad (\Gamma-1)\sum_{i=1}^k (2a_i + b_i + 1). \tag{5}$$

On the other hand, by the definition of clusters,

$$e(S_i) > (\Gamma - 1)\lfloor \frac{2a_i + 1 + 2a_{i+1} + 1 + 2b_i + 1}{2} \rfloor = (\Gamma-1)(a_i + a_{i+1} + b_i + 1),$$

which implies

$$(\Gamma-1)\sum_{i=1}^k (2a_i + b_i + 1) < \sum_{i=1}^k e(S_i). \tag{6}$$

Inequality (6) contradicts inequality (5), proving the correctness of the theorem. ∎

Theorem 2 *For every cluster S of a multigraph with $\Gamma > \Delta + 1$,*

$$|S| \leq \frac{\Gamma - 3}{\Gamma - 1 - \Delta}.$$

Proof. Express inequality (2) with respect to $|S|$ and then use $0 \leq \delta(S) + \nabla(S)$. ∎

3 Lower bounds

Although, for some multigraphs, Ω is a stronger lower bound for χ' than Γ, it turns out that is it not stronger than Δ and Γ combined.

Lemma 5 *For any multigraph G,*

$$\max(\Gamma(G), \Delta(G)) \leq \Omega(G) \leq \chi'(G). \tag{7}$$

Proof. For any edge-coloring of G and any $F \subseteq E(G)$, the number of edges colored the same color does not exceed $m(H) \leq \lfloor |V(H)/2 \rfloor$, where $H = G[F]$. Hence,

$$\chi'(H) \geq \lceil \frac{|F|}{m(F)} \rceil \geq \lceil \frac{e(H)}{\lfloor \frac{|V(H)}{2} \rfloor} \rceil.$$

To complete the proof, notice that if x is a vertex of the maximal degree in G and F is the set of edges incident to x, then $m(F) = 1$, implying that $\Delta(G) \leq \Omega(G)$. ∎

Theorem 3 *For every multigraph G, max$(\Gamma(G), \Delta(G)) = \Omega(G)$.*

Proof. By Lemma 5, we only need to prove that $\max(\Gamma(G), \Delta(G)) \geq \Omega(G)$. Let F be a set of edges for which

$$\lceil \frac{|F|}{m(F)} \rceil = \Omega(G)$$

and let $H = G[F]$. If $m(F) \geq (|V(H)| - 1)/2$, the result follows immediately. Thus, we assume that

$$m(F) < \frac{|V(H)| - 1}{2} \text{ and } \Omega > \max(\Delta, \Gamma).$$

By Tutte's theorem ([12]), there is a subset $K \subseteq V(H)$ such that the number of odd connected components of $H - K$ is $q = k + n - 2m(F)$, where $n = |V(H)|$ and $k = |K|$.

Let $\{V_i, F_i\}_{i=1}^{q+t}$ be the connected components of $G - K$, where the first q of them are odd and the remaining t are even. Let $|V_i| = 2a_i + 1$, for $i \in [1, q]$ and $|V_i| = 2a_i$, for $i \in [q + 1, q + t]$. Let C_i be the set of edges of H with one endpoint in V_i and the other in K ($i \in [1, q + t]$). Finally, let $E(K)$ denote the set of edges in F with both end-points in K.

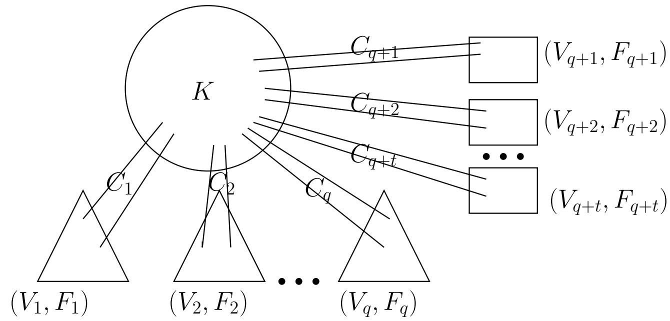

Figure 2: The triangles (resp. squares) represent odd (resp. even) components.

Using the assumption $\Omega > \max(\Delta, \Gamma)$, for every odd connected component,

$$\forall i \in [1, q], \quad |F_i| \leq \Gamma a_i \leq (\Omega - 1)a_i. \tag{8}$$

Since the maximum vertex degree is Δ,

$$\forall i \in [q+1, q+t], \quad |F_i| \leq \Delta a_i \tag{9}$$

$$|E(K)| + \sum_{i=1}^{q+t} |C_i| \leq 2|E(K)| + \sum_{i=1}^{q+t} |C_i| \leq \Delta\, k. \tag{10}$$

Combining (8), (9), (10), and using $\Omega - 1 \geq \Delta$ one more time, we get an upper bound for $|F|$:

$$|F| \leq |E(K)| + \sum_{i=1}^{q+t}(|F_i| + |C_i|) \leq \Delta k + (\Omega - 1)\sum_{i=1}^{q} a_i + \Delta \sum_{i=q+1}^{q+t} a_i \leq (\Omega - 1)(k + \sum_{i=1}^{q+t} a_i). \tag{11}$$

Since

$$n = \sum_{i=1}^{q}(2a_i + 1) + \sum_{i=q+1}^{q+t}(2a_i) + k,$$

we have

$$m(F) = \sum_{i=1}^{q+t} a_i + k.$$

Using this expression for $m(F)$ and inequality (11),

$$\Omega = \lceil \frac{|F|}{m(F)} \rceil \quad \leq \quad \frac{(\Omega - 1)(\sum_{i=1}^{q+t} a_i + k)}{\sum_{i=1}^{q+t} a_i + k} = \Omega - 1.$$

The contradiction disproves the assumption and completes the proof. ▮

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