# Toward computing $m(4)$

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WARNING: after the paper was published, M. Goldberg discovered a number of errors. Unfortunately, not all of them could be fixed using the method described in the paper. It is safe to claim that the proof to the bound $m_9(4) \geq 19$ is correct.

## 1 Introduction

A hypergraph is called bipartite [6] if its vertices can be colored in two colors so that no hyperedge is monochromatic. Erdös [5] defined a function $m(n)$ to be the minimal number of hyperedges in an $n$-uniform non-bipartite hypergraph. It is easy to see that $m(2) = 3$ and the corresponding hypergraph is a triangle. The Fano plane, the projective plane of order 2, shows that $m(3) \leq 7$; furthermore, it is not too difficult to prove that $m(3) = 7$.

The problem of calculating $m(n)$ becomes non-trivial with $n = 4$. Abbott and Hanson [1] constructed a hypergraph showing that $m(4) \leq 24$. Their construction was independently improved by Seymour[8] and Toft[10]. The new hypergraph[1] contains 23 quadruples of an 11-element set; thus $m(4) \leq 23$.

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[1] Both constructions turns out to be isomorphic, although it was not recognized at the time. To the authors' knowledge, the isomorphism of the hypergraphs was first established with the use of the software system *SetPlayer*[3].

The only known lower bound for $m(4)$ the authors were aware of was a trivial $m(4) \geq 8$. If $k$ is the cardinality of the vertex set of a hypergraph $G$, then there are $2^{k-1}$ different 2-colorings of $G$ and every hyperedge is monochromatic under $2^{k-4}$ of them. Thus, at least $2^{k-1}/2^{k-4} = 8$ hyperedges are needed to make $G$ non-bipartite. Obviously, the proof above is readily generalized to obtain $m(n) \geq 2^{n-1}$. (see [4].) Herzog and Schonheim [7] have shown that $m(n) \geq 2^n n/(n+2)$, implying $m(4) \geq 11$. Although much stronger lower bounds are known for asymptotics of $m(n)$ (see [2],[9]), they do not provide an improvement for small $n$. In this paper, we are concerned with the case of $n = 4$. Our main result is the following theorem.

**Theorem**. Let $m_k(4)$ denote the minimal number of hyperedges in a 4-uniform non-bipartite hypergraph with $k$-vertices. Then

$$m_k(4) \geq \begin{cases} 23 & \text{if } k = 9, 10; \\ 20 & \text{if } k = 11, 12; \\ 17 & \text{if } k = 13, 14. \end{cases}$$

We then use the theorem to prove

**Corollary**. $m(4) \geq 17$.

The theorem and the corollary are proved in Sections 3–5.

## 2 Terminology

The hypergraphs considered in the paper are always 4-uniform, that is, they are collections of quadruples (sets of size 4). The union of these sets is called the vertex set of the hypergraph and the sets are called hyperedges. Every coloring of a hypergraph is a 2-coloring of its vertex set. A coloring is called *balanced* (resp. *unbalanced*) *on a given set* $T$ *of vertices* if the number of vertices in $T$ colored in one of the colors differs from that for the other color by at most (resp. more than by) one. A coloring which is balanced (resp. unbalanced) on the vertex set of the hypergraph is simply called *balanced* (resp. *unbalanced*). If a coloring $\phi$ is monochromatic on a hyperedge $Q$, then we say that $Q$ blocks $\phi$. The set of colorings blocked by a given hyperedge $Q$ is denoted $C(Q)$; the *blocking degree* of a hyperedge $Q$ with respected to a given set $\mathcal{S}$ of colorings is the number $bdeg(Q; \mathcal{S}) = |\mathcal{S} \cap C(Q)|$.

The bound $m(4) \geq 8$ can be improved if the set of all colorings is reduced to the set $\mathcal{B}$ of balanced colorings. Although $|\mathcal{B}| < 2^{n-1}$, the blocking degree of every hyperedge is proportionally even smaller. For example, if $n = 12$, then $|\mathcal{B}| = 462$ and $bdeg(Q, \mathcal{B}) = 28$, implying $m_{12}(4) \geq \lceil 462/28 \rceil = 17$. It appears that a further reduction of a similar kind is not possible. An improvment of the lower bounds can, in principle, be obtained through the use of the inclusion-exclusion technique. Unfortunatelly, as often is the case, the analysis of the formula produced when using the technique is not tractable.

Our approach involves an analysis of pairwise intersections of the quadruples. We repeatedly branch the problem into subproblems according to the sizes of the hyperedge pairwise intersections. In one case, the hyperedge intersections imply a sizable overlap of the sets of colorings blocked by the hyperedges, and this can be used to increase the lower bound. In the other case, we define a certain subset $\mathcal{S} \subset \mathcal{B}$ and prove that the reduction in the size of the set of colorings is still smaller than that in the blocking degrees. This leads to the same lower bound as in the first case.

We say that two hyperedges $Q'$ and $Q''$ are *distant* if $|Q' \cap Q''| = 1$ or 2; otherwise, the hyperedges are called *close*. Given a hypergraph $\mathcal{H}$, we define an auxiliary graph $G(\mathcal{H})$ whose vertices are hyperedges of $\mathcal{H}$, and two vertices are adjacent iff the hyperedges are close. A hyperedge which is an isolated vertex in $G(\mathcal{H})$ is also called *isolated*. Given an order of the hyperedges in a hypergraph, we define the *actual* degree of a hyperedge $Q$ to be the number of colorings blocked by $Q$ and not blocked by any preceeding hyperedge. We assume that the vertices of $G(\mathcal{H})$ (equivalently, the hyperedges in $\mathcal{H}$) are always ordered in such a way that for every component of $G(\mathcal{H})$, every vertex, except for the first, is adjacent to at least one preceeding vertex of the component. We call such an ordering *standard*. A standard ordering implies that for every hyperedge which is not the first in its component, its actual degree is smaller than its blocking degree.

Throughout the paper, we formulate a number of lemmas. Most of them can be proved by straightforward, although sometimes tiresome checking; we leave the proofs of those lemmas to the reader.

## 3 The $k = 9, 10$ case.

Let $\mathcal{B}$ be the set of all balanced coloring of the set $\{1, 2, \ldots, k\}$, where $k = 9$ or 10. Then it is straightforward to compute that $|\mathcal{B}| = 126$ and the blocking degree of every quadruple is 6. This implies $m_4(k) \geq 21$. Given two quadruples $Q_1$ and $Q_2$, the size of the intersection $|C(Q_1) \cap C(Q_2)|$ is determined by the value of $|Q_1 \cap Q_2|$, as the following "blocking table" shows.

| $\lvert Q_i \cap Q_j \rvert$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\lvert C(Q_i) \cap C(Q_j) \rvert$ | 2 | 0 | 0 | 1 |

**Lemma 1.** Any collection of eight quadruples of a set with at most ten elements contains two quadruples with at least two elements in common.

**Proof.** Indeed, the number of pairs that are covered by the quadruples (counting multiple coverings as well) equals $8 \times 6 = 48$. On the other hand, the number of pairs in a 10-element set is 45. Thus, at least one pair is covered by at least two quadruples. ■

Let $\mathcal{H} = \{Q_1, Q_2, \ldots, Q_m\}$ be a non-bipartite hypergraph on $k$ vertices ($k = 9, 10$). We consider two cases.

**Case 1:** $\exists$ at most 7 isolated quadruples.

Assume that the hyperedges are ordered in the standard way. Then, every hyperedge which is not the first in its component of $G(\mathcal{H})$ blocks at most 5 colorings. Since there are at most 7 isolated hyperedges, at least half of the remaining $m-7$ hyperedges block $\leq 5$ colorings. Thus, we have $7 \times 6 \ + \ (m-7) \times \frac{11}{2} \geq 126$ implying $m \geq 23$.

**Case 2:** the number of isolated quadruples of the hypergraph is at least 8.

Using lemma 1 we find two isolated hyperedges with two elements in common. Let the hyperedges be $Q_1 = (1,2,3,4)$ and $Q_2 = (1,2,5,6)$ and let $\mathcal{S}$ be a subset of $\mathcal{B}$ comprised of all colorings that are unbalanced on both $Q_1$ and $Q_2$. Furthemore, let $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ be a partitioning of $\mathcal{S}$ defined by:

$\mathcal{S}_1$**:** the set of colorings unbalanced on $Q_1 \cap Q_2$;

$\mathcal{S}_2$**:** the set of colorings balanced on $Q_1 \cap Q_2$ and on $V - (Q_1 \cup Q_2)$;

$\mathcal{S}_3$**:** the set of colorings balanced on $Q_1 \cap Q_2$ and unbalanced on $V - (Q_1 \cup Q_2)$.

Notice that if a coloring is unbalanced on a set with at most three elements, then it is monochromatic on this set.

**Lemma 2.** $|\mathcal{S}| = 30$, $|\mathcal{S}_1| = 16$, $|\mathcal{S}_2| = 12$, $|\mathcal{S}_3| = 2$. ∎

Define the type of a quadruple $Q$ to be a triple $(a,b,c)$ where $a = |Q \cap \{1,2\}|$; $b = |Q \cap \{3,4\}|$, $c = |Q \cap \{5,6\}|$. For the case under consideration, $1 \leq a+b \leq 2$ and $1 \leq a+c \leq 2$. It is easy to check, that for any quadruple, its blocking degrees with respect to sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ depend on the type of the quadruple only; in addition, any two quadruples whose types are $(a,b,c)$ and $(a,c,b)$ have the same degrees. The table below shows the dependence of the blocking degrees on the type of a quadruple; the empty entries in the table correspond to 0's.

| TYPE | (2,0,0) | (1,1,1) | (0,2,2) | (1,1,0) | (0,2,1) | (1,0,0) | (0,1,1) |
|---|---|---|---|---|---|---|---|
| SAMPLE | 1 2 7 8 | 1 3 5 7 | 3 4 5 6 | 1 3 7 8 | 3 4 5 7 | 1 7 8 9 | 3 5 7 8 |
| $\mathcal{S}_1$ | | 1 | | | | | 2 |
| $\mathcal{S}_2$ | | | | 1 | | | |
| $\mathcal{S}_3$ | | | 2 | | | 1 | |

Thus, from the table we see that the colorings from sets $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ are blocked by quadruples of different types. To block $\mathcal{S}_1$ we need at least 8 quadruples (the types are either $(1,1,1)$ or $(0,1,1)$); to block colorings from $\mathcal{S}_2$ we need 12 quadruples of type $(1,1,0)$, and to block colorings from $\mathcal{S}_3$, we need at least one quadruple. It implies that at least 23 quadruples[2] are needed.

### 3.1 The $k = 11, 12$ case.

For this case, the set $\mathcal{B}$ of all balanced colorings contains 462 elements and the blocking degree of each quadruple is 28. Thus, the initial lower bound is $m_k(4) \geq \lceil 462/28 \rceil = 17$. The blocking table is as follows.

| $\lvert Q_i \cap Q_j \rvert$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\lvert C(Q_i) \cap C(Q_j) \rvert$ | 6 | 0 | 1 | 7 |

As before, $\mathcal{H} = \{Q_1, Q_2, \ldots, Q_m\}$ is a non-bipartite hypergraph and $A_1, \ldots, A_p$ are the components of $G(\mathcal{H})$.

**Case 1:** $G(\mathcal{H})$ contains at least one isolated vertex.

Let $Q_1 = \{1, 2, 3, 4\}$ be an isolated hyperedge, and let $\mathcal{S}_1$ be the set of all balanced colorings that are unbalanced on $Q_1$.

**Lemma 3.** For every hyperedge $Q_i (i = 2, \ldots m)$, if $|Q_i \cap Q_1| = 1$ (resp. $|Q_i \cap Q_1| = 2$) then $bdeg(Q_i; \mathcal{S}_1) = 13$ (resp. $bdeg(Q_i; \mathcal{S}_1) = 12$). ■

The bound $m \geq 19$ follows immediately. To prove a stronger bound, we branch the case into two subcases.

**Case 1.1:** there is at least one more isolated hyperedge in $\mathcal{H}$.

Let $Q_2$ be an isolated hyperedge different from $Q_1$ and let $\mathcal{S}_2 \subset \mathcal{S}_1$ be the set of colorings in $\mathcal{S}_1$ unbalanced on both $Q_1$ and $Q_2$.

**Lemma 4.** If $|Q_1 \cap Q_2| = 1$, then $|\mathcal{S}_2| = 106$ and $\forall Q' \in \mathcal{H}, Q' \neq Q_1, Q_2, bdeg(Q'; \mathcal{S}_1) \leq 6$. If $|Q_1 \cap Q_2| = 2$, then $|\mathcal{S}_2| = 112$, $bdeg(1, 2, 5, 6) = 12$, and $\forall Q' \in \mathcal{H}(Q' \neq Q_1, Q_2, (1, 2, 5, 6))$, $bdeg(Q'; \mathcal{S}_1) \leq 6$.

[2] A more careful analysis shows that to block the colorings from $\mathcal{S}_1$, we need 12 quadruples; this gives the lower bound of 27 for the case.

Thus, from the previous lemma, depending on the size of $Q_1 \cap Q_2$, we have either $12+6\times(m-3) \geq 112$, or $6\times(m-2) \geq 106$, yielding $m \geq 20$. cases. ■

**Case 1.2:** $Q_1$ is the only isolated hyperedge.

We call a pair of quadruples $Q', Q''$ *special* if $Q' \cap Q'' = \emptyset$ and $|Q' \cap Q_1| = |Q'' \cap Q_1| = 2$.

**Lemma 5.** If $Q'$ and $Q''$ belong to the same component of $G(\mathcal{H})$, then there are at least two colorings in $\mathcal{S}_1$ blocked by both hyperedges, unless $(Q', Q'')$ is a special pair, in which case the number of colorings blocked by both hyperedges is 0. ■

**Lemma 6.** For every component $A'$ of $G(\mathcal{H})$, $\sum_{Q' \in A'} adeg(Q') \leq 12|A'|$.

**Proof:** If $A'$ contains a hyperedge $Q'$ intersecting $Q_1$ in two elements, then we place $Q'$ first in the component. Obviously, $adeg(Q') \leq 12$ and by Lemma 4, for every other hyperedge $Q'' \in A'$, if $bdeg(Q'') = 13$, then $adeg(Q'') \leq 11$.

If $A''$ is a component without hyperedges intersecting $Q_1$ in two elements, then applying lemma 4 again we obtain

$$\sum_{Q' \in A''} adeg(Q') \leq 13 + 11(|A''| - 1) \leq 12|A''|. \quad ■$$

Completing case 1.1, as well as the whole case 1, is now trivial:

$$224 \leq \sum_{i=2}^{p} 12|A_i| = 12(m-1) \quad \textit{and} \quad m \geq 1 + \lceil 224/12 \rceil = 20.$$

**Case 2:** Every component of $G(\mathcal{H})$ contains at least two vertices.

First, we prove that $m \geq 19$.

**Lemma 7.** Let $Q'$ be a hyperedge which is not numbered first in its component of $G(\mathcal{H})$. Then $adeg(Q') \leq 22$. ■

Indeed, observe that the number of components $p \leq m/2$ and use the blocking table to prove Now,

$$462 \leq \sum_{i=1}^{p} adeg(Q_i) \leq \sum_{i=1}^{p} bdeg(Q_i) = (28 + 22(|A_i| - 1)) = 22m + 6p. \qquad (*)$$

But, for the case under consideration, $p \leq m/2$, therefore $462 \leq 25m$, and $m \geq 19$.

To improve the bound, we need two more auxiliary statements.

**Lemma 8.** Let $P$ and $Q$ be two disjoint quadruples and let $\mathcal{S}$ be the set of balanced colorings, that are unbalanced on both $P$ and $Q$. If $R$ and $S$ are close quadruples both distant from $P$ and $Q$, then $bdeg(R;\mathcal{S}) \leq 6$, and at least one coloring in $\mathcal{S}$ is blocked by $R$ and by $S$. ■

**Lemma 9.** Let $(R_1, S_1), (R_2, S_2), \ldots, (R_8, S_8)$ be 8 pairs of distinct quadruples of a set with $\leq$ 12 elements. Then, there are 8 pairs of elements each covered by two quadruples from different pairs.

**Proof.** For every $i = 1, \ldots 8$, the quadruples of the pair $(R_i, S_i)$ cover at least 9 pairs of elements, making the total number of covered pairs of elements 72. On the other hand, since the cardinality of the ground set $\leq 12$, the number of sets with two elements is at most 66. This proves the lemma. ■

Now we proceed to proving $m \geq 20$ in case 2. Let us assume, that the bound is wrong and $m = 19$. Then, from[3]

$$p \leq m/2 \quad and \quad 22m + 6p \leq 462$$

it follows that $8 \leq p \leq 9$. It is easy to see then that for the sizes $\{|A_i|\}$ $(i = 1, \ldots, p)$ the only possibilities, up to reordering, are

$$(5, 2, 2, 2, 2, 2, 2, 2), (4, 3, 2, 2, 2, 2, 2, 2), \quad and \quad (3, 2, 2, 2, 2, 2, 2, 2, 2).$$

Let there be a component consisting of two disjoint hyperedges. Then we consider the set $\mathcal{S} \subset \mathcal{B}$ of colorings that are unbalanced on both hyperedges of the components. Using lemma 6 we have

$$|\mathcal{S}| \leq \sum_{i=1}^{p}(6 + 5(m_i - 1)) = 5m + p \leq 5 \times 19 + 9 = 89.$$

But it is easy to compute that $|\mathcal{S}| = 112$, which proves that every component of size two consists of hyperedges intersecting in three elements. Now we can easily reject $(5, 2, 2, 2, 2, 2, 2, 2)$ and $(4, 3, 2, 2, 2, 2, 2, 2)$ as possible lists of components' sizes. For example, for the latter one (similar for the former), it must have been, that

$$462 \leq (28 + 3 \times 22) + (28 + 2 \times 22) + (28 + 21) \times 6 = 460.$$

Thus, if $m = 19$, then $G(\mathcal{H})$ has 9 components, one is of size 3, and the rest are of size 2. We apply lemma 7 to find 8 pairs of elements that are covered by hyperedges from different components. From the blocking table, we see that two quadruples intersecting

[3]see the inequality $(*)$

in two elements block 1 coloring of $\mathcal{B}$ in common. Therefore, for at least 4 components $\{A_i\}$ of size two, the value $\sum_{Q\in A_i} adeg(Q)$ is smaller than $\sum_{Q\in A_i} bdeg(Q)$, implying

$$\sum_{i=1}^{p}\sum_{Q\in A_i} adeg(Q_i) \le \sum_{i=1}^{m} bdeg(Q_i) - 4 \le (28+2\times 22)+(28+21)\times 6) - 4 = 460,$$

which is impossible since $|\mathcal{S}| = 460$. This completes the analysis for $k = 11, 12$.

## 4 The $k = 13, 14$ case.

There are 1716 balanced colorings on 13 or 14 vertices, and each quadruple blocks 120 of them. A lower bound $m \ge 15$ follows immediately. The "blocking table" in this case is as follows.

| $\lvert Q_i \cap Q_j\rvert$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\lvert C(Q_i)\cap C(Q_j)\rvert$ | 20 | 1 | 8 | 36 |

**Lemma 10.** Let $Q'$ and $Q''$ be two quadruples distant from quadruple $Q_1$ and let $\mathcal{S}$ be the set of coloring in $\mathcal{B}$ that are unbalanced on $Q_1$. If no coloring in $\mathcal{S}$ is blocked by both $Q'$ and $Q''$, then $(Q'\cap Q_1)\ \cap\ (Q''\cap Q_1) = \emptyset$ and $|Q'\cap Q_1| = |Q''\cap Q_1| = 2$. ■

Again, $\mathcal{H} = \{Q_1,\ldots,Q_m\}$ is a non-bipartite hypergraph and $A_1,\ldots,A_p$ are the components of $G(\mathcal{H})$.

**Case 1:** there is at least one isolated quadruple, say $Q_1 = (1,2,3,4)$.

Let $\mathcal{S}$ be the subset of $\mathcal{B}$ consisting of colorings unbalanced on $Q_1$. Then $|\mathcal{S}| = 840$ and each quadruple $Q_i$ blocks 56 colorings in $\mathcal{S}$. However, from the lemma we conclude that there can be at most six hyperedges whose actual degree is 56; for any other hyperedge its actual degree $\le 55$. Therefore, $56\times 6\ +\ 55\times(m-7) \ge 840$, or $m \ge 18$.

**Case 2:** every component of $\mathcal{H}$ contains at least two hyperedges.

Call a component type 1 if it is comprised of two disjoint hyperedges; any other component is called type 2.

**Lemma 11.** If $A_i$ is a type 1 component, then $\sum_{Q\in A_i} bdeg(Q) = 220$; otherwise $\sum_{Q\in A_i} bdeg(Q) < 107|A_i|$. ■

Now, if every component of $G(cH)$ is type 2, then

$$1716 \le \sum_i \sum_{Q\in A_i} bdeg(Q) < 107m \quad \text{and} \quad m \ge 17.$$

If $G(\mathcal{H})$ has a type 1 component, let $Q_1 = (1,2,3,4)$ and $Q_2 = (5,6,7,8)$ be the hyperedges of that component and let $\mathcal{S}$ be the subset of $\mathcal{B}$ consisting of all colorings that are unbalanced on both $Q_1$ and $Q_2$. Then $|\mathcal{S}| = 416$ and every hyperedge blocks at most 28 colorings[4] in $\mathcal{S}$. We have $28(m-2) \geq 416$, or $m \geq 17$. ■

**Proof of the corollary.** Suppose, $m(4) < 17$ and let $\mathcal{H}$ be a non-bipartite 4-uniform hypergraph with 16 or fewer hyperedges which has the minimal number of vertices. Obviously, the vertex number is at least 15. We easily compute that the number of pairs of the vertices is $> \binom{15}{2} = 105$. On the other hand, the total number of pairs in all hyperedges is $\leq 6 \times 16 = 96$. Thus, at least one pair of vertices does not belong to any hyperedge. Therefore, if we contract this pair of vertices into a new vertex, we obtain a new non-bipartite hypergraph with the same number of hyperedges, but fewer number of vertices. This contradicts the minimality of $\mathcal{H}$. ■

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[4] To check the latter, remember that every hyperedge distinct from $Q_1$ and $Q_2$ meets both these hyperedges in one or two elements.

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