

On the Minimal Cut Problem

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ABSTRACT

This paper is devoted to the problem of partitioning the vertex set of a graph into two equally sized subsets (to within one element) in such a way that the number of edges cut is minimal. A sequence of $n \log n$ - time approximation algorithms is presented. It is proved that for almost all graphs each member of the sequence produces a partition close to the best solution.

1. Introduction

This paper is devoted to the problem of partitioning the vertex set of a graph $G(V, E)$ into two disjoint subsets V_1 and V_2 in such a way that

$$(a) |V_1| = \lfloor \frac{1}{2}|V| \rfloor, |V_2| = \lceil \frac{1}{2}|V| \rceil;$$

(b) the number of the edges that have one end in V_1 and one end in V_2 is minimal.

The problem and its diverse generalizations were studied in [1], [5], [9], [10]. It is known [6] that the problem is *NP*-complete, making unlikely the existence of a polynomial algorithm producing an exact solution for every graph. However, in the areas of its practical applications such as computer program segmentation [8], or layout of electronic circuits on computer boards ([3], [4], [7]), the problem is particularly important for graphs on many thousands of vertices. In such a situation, one cannot afford even a quadratic algorithm.

In this paper we present a sequence of $n \log n$ -algorithms that produce good approximate solutions. It is proved that for almost all graphs without parallel edges, every algorithm of the sequence constructs a partition close to the best solution.

Throughout the paper the following notations will be used. Given

a graph G , $V = V(G)$ and $E = E(G)$ denote respectively the set of vertices and the set of edges of G ; $n(G) = |V(G)|$, $p(G) = |E(G)|$. We permit graphs to have parallel edges; however, loops are not allowed. If $x, y \in V(G)$ then $d(x)$ is the degree of x ; $d(x, y)$ is the number of edges joining x and y . If $A, B \subset V$ then $p(A, B)$ denotes the number of edges of the form $(x, y) (x \in A, y \in B)$.

We will consider only those partitions of $V(G)$ that satisfy the condition (a) above. It is clear that if $n = n(G)$ is even (resp. odd) then the number of different partitions is $\frac{1}{2} \binom{n}{\frac{n}{2}}$ (resp. $\binom{n}{\frac{n-1}{2}}$). If σ is a partition of $V(G)$ then $c_\sigma(G)$ denotes the number of edges cut by σ ; $c(G) = \min_\sigma c_\sigma(G)$, $c(n, p) = \max_G c(G)$ where G ranges over the set of all graphs with n vertices and p edges.

Let $G(n, p) = \{G \mid n(G) = n, p(G) = p\}$ where G has no parallel edges. Then $K(n, p) = |G(n, p)|$; $K(n, p, \alpha) = |\{G: G \in G(n, p) \text{ and } c(G) \leq \alpha p(G)\}|$.

All concepts we do not explain here can be found in [2].

2. Upper and Lower Bounds

We start with the following simple bound

THEOREM 1. For every graph $G(V, E)$ with $|V| = n$ and $|E| = p$

$$c(G) \leq \begin{cases} \frac{p \cdot n}{2(n-1)} & \text{if } n \text{ is even,} \\ \frac{p(n+1)}{2n} & \text{if } n \text{ is odd} \end{cases}$$

PROOF. First let n be even. Then

$$\frac{1}{2} \binom{n}{\frac{n}{2}} c(G) \leq \sum_{\sigma} c_{\sigma}(G) \quad (*)$$

where σ ranges over the set of all partitions of G .

Straightforward calculations show that in the right side of (*) the contribution of each edge is exactly $\binom{n-2}{\frac{n}{2}-1}$, therefore

$$\frac{1}{2} \binom{n}{\frac{n}{2}} c(G) \leq \binom{n-2}{\frac{n}{2}-1} p(G)$$

The result follows.

In the case of odd n instead of (*) we have

$$\binom{\frac{n}{2}}{\frac{n-1}{2}} c(G) \leq \sum_a c(G)$$

but now the contribution of each edge is equal to $2 \binom{\frac{n-2}{2}}{\frac{n-1}{2}-1}$ resulting in

$$c(G) \leq \frac{1}{2} \left(1 + \frac{1}{n}\right) p.$$

A lower bound for $c(n, p)$ is derived by considering the following graph $G_{n,p}^*$.

Let s be the largest integer such that $p \geq s(n-1) - \frac{s(s-1)}{2}$ and let $r = p - s(n-1) + \frac{s(s-1)}{2}$. It can be easily proved that $0 \leq r \leq n-s-2$. Now, the set of vertices V and the set of edges E of $G_{n,p}^*$ are defined by

$$V = \{x_1, x_2, \dots, x_n\}$$

$$E = \{(x_i, x_j) \mid i = 1, \dots, s; j = 1, \dots, n(i \neq j)\} \cup$$

$$\{(x_{s+1}, x_j) \mid j = s+2, \dots, s+r+1\}.$$

Thus, if $r = 0$ then $G_{n,p}^*$ contains exactly s vertices of degree $n-1$ and their deletion produces an empty graph on $n-s$ vertices. If $r > 0$ then $G_{n,p}^* - \{x_1, \dots, x_s\}$ is a graph with $n-s$ vertices and r edges $(x_{s+1}, x_{s+2}), \dots, (x_{s+1}, x_{s+r+1})$.

Routine calculations show that the minimal partition cuts the set $\{x_1, \dots, x_s\}$ into two parts of sizes $\lfloor \frac{s}{2} \rfloor$, $\lceil \frac{s}{2} \rceil$ and

$$c(G_{n,p}^*) = \left\{ \frac{s(2n-s)}{4} \right\} + (r - \lceil \frac{(n-s)/2}{2} \rceil - 1)^+$$

where $\{z\}$ denotes the integer nearest to z and $(z)^+ = \frac{z + |z|}{2}$.

Since $p = \frac{s(2n-s)}{2} - \frac{s-2r}{2}$ one can see that $G_{n,p}^*$ provides a

lower bound for $c(n, p)$ that is very close to the upper bound proved above.

CONJECTURE. For all n and p

$$c(n, p) = c(G_{n, p}^*).$$

From the proof of theorem 1 it clearly follows that if $c(G)$ is close to $\frac{1}{2}p$ then for most partitions σ the value $c_\sigma(G)$ is close to $\frac{1}{2}p$ as well. It may seem from this that such graphs are rather rare. However, this is not the case. As was proved in [8] if $p/n \rightarrow \infty$ almost all graphs are such that for every partition $c_\sigma(G)$ is asymptotically $\frac{1}{2}p$. As a matter of fact, in [8] one can find essentially stronger results. The next theorem can be considered as an illustration of them. We give here an independent and simple proof of the theorem.

THEOREM 2. For any positive $\alpha < \frac{1}{2}$ there exists a β such that if $p > \beta n$ and $n \rightarrow \infty$ then

$$\frac{K(n, p, \alpha)}{K(n, p)} \rightarrow 0.$$

PROOF. Let us fix some set V of n vertices and some partition $\sigma = (V_1, V_2)$. Also, let $R(n, p, i)$ (resp. $R_\sigma(n, p, i)$) denote the number of graphs on V with p edges and $c(G) = i$ (resp. $c_\sigma(G) = i$). Then $K(n, p, \alpha) = \sum_{i \leq \alpha p} R(n, p, i) < 2^n \sum_{i \leq \alpha p} R_\sigma(n, p, i)$.

Let us assume that n is even (the odd n case is similar). Then

$$K(n, p, \alpha) < 2^n \cdot \sum_{i \leq \alpha p} \binom{\frac{n^2}{4}}{i} \cdot \binom{\frac{n^2}{4} - \frac{n}{2}}{p - i}$$

Since for any $i < \frac{1}{2}p$

$$\binom{\frac{n^2}{4}}{i-1} \cdot \binom{\frac{n^2}{4} - \frac{n}{2}}{p-i+1} < \binom{\frac{n^2}{4}}{i} \cdot \binom{\frac{n^2}{4} - \frac{n}{2}}{p-i},$$

$$\frac{K(n, p, \alpha)}{K(n, p)} < \frac{p \cdot 2^n \binom{\frac{n^2}{4}}{q} \binom{\frac{n^2}{4} - \frac{n}{2}}{p - q}}{\binom{\frac{n^2}{2} - \frac{n}{2}}{p}}, \text{ where } q = \lfloor \alpha p \rfloor.$$

Then

$$\frac{\binom{\frac{1}{4}n^2}{q} \binom{\frac{1}{4}n^2 - \frac{1}{2}n}{p - q}}{\binom{\frac{1}{2}n^2 - \frac{1}{2}n}{p}} = \frac{\binom{\frac{1}{4}n^2}{q} \binom{\frac{1}{4}n^2}{p - q}}{\binom{\frac{1}{4}n^2}{\frac{1}{2}p}}.$$

$$\frac{\binom{\frac{1}{4}n^2}{\frac{1}{2}p}}{\binom{\frac{1}{2}n^2}{p}} \cdot \frac{\binom{\frac{1}{2}n^2}{p} \binom{\frac{1}{4}n^2 - \frac{1}{2}n}{p - q}}{\binom{\frac{1}{2}n^2 - \frac{1}{2}n}{p} \binom{\frac{1}{4}n^2}{p - q}}$$

$$= \frac{1}{2^{p-q}} \prod_{i=1} \left(\frac{q+i}{\frac{1}{4}n^2 - q + 1 - i} \cdot \frac{\frac{1}{4}n^2 - \frac{1}{2}p + 1 - i}{\frac{1}{2}p + i} \right) \cdot \frac{\binom{p}{\frac{1}{2}p} \binom{\frac{1}{2}n^2 - p}{\frac{1}{4}n^2 - \frac{1}{2}p}}{\binom{\frac{1}{2}n^2}{\frac{1}{4}n^2}}.$$

$$\begin{aligned}
& \prod_{j=1}^{\frac{n}{2}} \left(\frac{\frac{1}{2}n^2-j+1}{\frac{1}{4}n^2-j+1} \cdot \frac{\frac{1}{4}n^2-p+q-j+1}{\frac{1}{2}n^2-p-j+1} \right) \\
& < \prod_{i=1}^{\frac{1}{2}p-q} \left(\frac{q+i}{\frac{1}{2}p+i} \cdot \frac{\frac{1}{4}n^2-\frac{1}{2}p+1-i}{\frac{1}{4}n^2-q+1-i} \right) \cdot \prod_{j=1}^{\frac{n}{2}} \frac{\frac{1}{2}n^2-j+1}{\frac{1}{4}n^2-j+1} < \left(\frac{\frac{1}{2}p}{p-q} \right)^{\frac{1}{2}p} \\
& 2^{\frac{n}{2}} \left(1 + \frac{1}{n} \right)^{\frac{n}{2}} < \sqrt{e} \cdot 2^{\frac{n}{2}} \cdot \left(\frac{1}{2(1-\alpha)} \right)^{p(\frac{1}{2}-\alpha)}
\end{aligned}$$

The result follows for $p > \beta n$, where β is such that

$$2^{\frac{3}{2}} \cdot \left(\frac{1}{2(1-\alpha)} \right)^{\beta(\frac{1}{2}-\alpha)} < 1.$$

3. Algorithms

We now describe a sequence $\{Cut(2k)\}$ of $n \log n$ algorithms that produces approximate solutions to the problem. Each algorithm of the sequence reduces the problem for the original graph to that for a smaller graph and does so several times until the final graph is sufficiently small to be treated by an exhaustive search procedure. We do not specify this procedure. Let S_{2k} denote an arbitrary algorithm which for every input $[G(V, E); A; B]$ where $|V| \leq 2k$, A and B are disjoint subsets of V and $|A|, |B| \leq \frac{1}{2}|V|$, produces a partition $\tau = (V_1, V_2)$ such that $A \subset V_1$, $B \subset V_2$ and, for which, with these restrictions, the number of the edges cut by τ is minimal. Since k is assumed to be bounded the running time of S_{2k} can be taken as a constant.

We start with the descriptions of two procedures.

Procedure CONTRACTION ($2k$)

Input: Graph $G(V, E)$ on $2kq$ vertices (k, q are integers)

Output: Graph $G'(V', E')$ on $2q$ vertices.

1. Let x_1, x_2, \dots, x_{2kq} be the order of the vertex set of G . For subgraphs G_1, G_2, \dots, G_q induced on subse

- $X_1 = (x_1, \dots, x_{2k}),$
 $X_2 = (x_{2k+1}, \dots, x_{4k}), \dots, X_q = (x_{2kq-2k+1}, \dots, x_{2kq}).$
2. Apply S_{2k} to $[G_i; \emptyset; \emptyset]$. Let $\tau_i = (Y_{2i-1}, Y_{2i})$ be the resulting partition of G_i ($i = 1, 2, \dots, q$).
 3. Form graph $G'(V', E')$ on $2q$ vertices x_1', \dots, x_{2q}' defining two vertices x_i' and x_j' to be joined by m edges iff there are exactly m edges between Y_i and Y_j . Stop. \square

Procedure EXTENSION

Input: Graph $G(V, E)$; ordered set $R = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_r\}$ ($R \subset V; r \leq 2k-1$); partition (\bar{V}_1, \bar{V}_2) of $G-R$.

Output: Partition (V_1, V_2) of G .

1. If r is even go to 5 otherwise proceed to 2.
2. Put $H = G - \{\bar{x}_1\}$.
3. Apply EXTENSION to H ; let (\bar{U}_1, \bar{U}_2) be the resulting partition of H .
4. Calculate $d(\bar{x}_1, \bar{U}_1)$ and $d(\bar{x}_1, \bar{U}_2)$. If $d(\bar{x}_1, \bar{U}_1) \leq d(\bar{x}_1, \bar{U}_2)$ then put $V_1 = U_1, V_2 = U_2 \cup \{x_1\}$, otherwise put $V_1 = U_1 \cup \{x_1\}, V_2 = U_2$. Stop.
5. Form graph F on the vertices $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_r, \bar{x}_{r+1}, \bar{x}_{r+2}$ by contracting subsets \bar{V}_1, \bar{V}_2 into $\bar{x}_{r+1}, \bar{x}_{r+2}$ respectively.
6. Apply S_{2k} to $[F; \{\bar{x}_{r+1}\}; \{\bar{x}_{r+2}\}]$. Let (W_1, W_2) be the resulting partition of F .
7. Construct partition (V_1, V_2) of G from (W_1, W_2) by substituting \bar{V}_1 and \bar{V}_2 for \bar{x}_{r+1} and \bar{x}_{r+2} respectively. Stop. \square

Algorithm CUT ($2k$)

Input: Graph $G(V, E)$.

Output: Partition $\tau(V_1, V_2)$ of G ; the number c of the edges cut by τ .

1. Put $H = G(V, E)$.
2. If $n(H) \leq 2k$ apply S_{2k} . Stop.
3. Calculate the remainder r of $n = v(H)$ modulo $2k$. If $r > 0$ go to 4 otherwise go to 7.
4. Construct sequence $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_r$ of vertices by the rule:
 - (i) \bar{x}_1 is a vertex of the maximal degree; (ii) \bar{x}_{i+1} is such that $d^*(\bar{x}_{i+1}) - d(\bar{x}_i, \bar{x}_{i+1})(n-i-1)$ is maximal; where $d^*(\bar{x}_{i+1})$ is

the degree of \bar{x}_{i+1} in $F = H - \{x_1, \dots, x_{i-1}\}$.

5. Apply CUT(2k) to $H' = H - \{\bar{x}_1, \dots, \bar{x}_r\}$; let τ be the resultant partition.
6. Apply EXTENSION to $[H; \bar{x}_1, \bar{x}_2, \dots, \bar{x}_r; \tau]$. Stop.
7. Apply CONTRACTION(2k). Let \bar{H} be the output.
8. Apply CUT(2k) to \bar{H} ; let $\bar{\tau}$ be the resulting partition.
9. Restore the partition τ of H from $\bar{\tau}$ be the resulting partition.
10. Restore the partition τ of H from $\bar{\tau}$ by replacing vertices of i with the corresponding sets of H . Print. Stop.

4. Running Time and Accuracy

It is not hard to see that the running time of CUT(2k) is $O(n \log n)$ for every fixed k . Consider:

- 1) Step 4 in CUT(2k) can be performed in linear time.
- 2) CUT (2 k) calls itself recursively, each time on a graph half size of the previous. Therefore, $\log n$ calls are made to CUT(2k).
- 3) EXTENSION and CONTRACTION (2k) are evidently linear.

DEFINITION. A partition τ of a graph G with n vertices and p edge is called admissible if

$$c_{\tau}(G) \leq \begin{cases} \frac{pn}{2(n-1)} & \text{if } n \text{ is even} \\ \frac{p(n+1)}{2n} & \text{if } n \text{ is odd} \end{cases}$$

LEMMA. Let the input partition of EXTENSION be admissible and let the sequence $\bar{x}_1, \dots, \bar{x}_r$ of vertices be chosen according to the rule from Step 4 of CUT(2k). Then the output partition is also admissible.

PROOF. It is sufficient to consider the cases $r = 1$ and r even. We will prove the case r even only since the case $r = 1$ is much simpler and can be done in a similar way.

Let us consider $\frac{1}{2}r$ graphs defined by

$$G_1 = G, G_{i+1} = G_i - \{\bar{x}_{2i-1}, \bar{x}_{2i}\} \quad (i = 1, \dots, \frac{r}{2} - 1).$$

From the rule in step 4 of CUT(2k) it follows that given G_i and $\bar{x}_{2i-1}, \bar{x}_{2i}$ maximizes the quantity $d^*(\bar{x}_{2i}) - d(\bar{x}_{2k}, \bar{x}_{2i-1})(n(G_i) - 2)$ in G_i , where $d^*(x)$ denotes the degree of x in the graph G_i .

Therefore if we prove the lemma in the case $r = 2$, then each of the graphs $G_{r/2}, \dots, G_2, G_1$ gets an admissible partition and therefore Step 6 in EXTENSION produces an admissible partition too.

Now, let $\{G(V, E); \bar{x}_1, \bar{x}_2; \sigma = (\bar{v}_1, \bar{v}_2)\}$ be the input of EXTENSION, let \bar{x}_2 be such that $d(\bar{x}_2) - d(\bar{x}_1, \bar{x}_2)(n-2)$ is maximal, and also let $\tau = (v_1, v_2)$ be the output partition.

Then

$$\begin{aligned} c_\tau &\leq c_\sigma(G - \{\bar{x}_1, \bar{x}_2\}) + d(\bar{x}_1, \bar{x}_2) + \frac{d(\bar{x}_1) - d(\bar{x}_1, \bar{x}_2) + d(\bar{x}_2) - d(\bar{x}_1, \bar{x}_2)}{2} \\ &\leq \frac{1}{2}p(G - \{\bar{x}_1, \bar{x}_2\}) \cdot \frac{n-2}{n-3} + \frac{d(\bar{x}_1) + d(\bar{x}_2)}{2} \\ &= \frac{1}{2}(p(G) - d(\bar{x}_1) - d(\bar{x}_2) + d(\bar{x}_1, \bar{x}_2)) \frac{n-2}{n-3} + \frac{d(\bar{x}_1) + d(\bar{x}_2)}{2} \\ &= \frac{1}{2}p(G) \cdot \frac{n}{n-1} + \frac{1}{2(n-3)} \left(\frac{2p(G)}{n-1} + d(\bar{x}_1, \bar{x}_2)(n-2) - d(\bar{x}_1) - d(\bar{x}_2) \right) \end{aligned}$$

Thus, the result would follow from the inequality

$$\frac{2p(G)}{n-1} + d(\bar{x}_1, \bar{x}_2)(n-2) - d(\bar{x}_1) - d(\bar{x}_2) \leq 0.$$

If it fails then $\forall x \neq x_1$

$$\frac{2p(G)}{n-1} + d(\bar{x}_1, x)(n-2) - d(\bar{x}_1) - d(x) > 0$$

due to maximality of $d(\bar{x}_2) - d(\bar{x}_1, \bar{x}_2)(n-2)$.

Therefore,

$$\frac{2p(G)}{n-1}(n-1) + (n-2) \sum_{x \neq x_1} d(\bar{x}_1, x) > (n-1)d(\bar{x}_1) + \sum_{x \neq x_1} d(x)$$

implying

$$2p(G) + (n-2)d(\bar{x}_1) > (n-1)d(\bar{x}_1) + 2p - d(\bar{x}_1)$$

which is not possible.

Now we are ready to prove the following.

THEOREM 3. If τ is the output partition of CUT(2k) applied to a graph $G(V, E)$ then τ is admissible.

PROOF. It will be carried out by induction on the number Θ of calls of the procedure $S(2k)$.

If $\Theta = 1$ then $n(G) \leq 2k$ and the result follows from Theorem 1.

Let $\Theta > 1$ and let H be the graph to which S_{2k} was applied the last time. The number $n(H)$ is apparently $\leq 2k$. Because of the lemma we can assume that H is the last term of a sequence of graphs $F_1, F_2, \dots, F_t = H$ such that F_{i+1} is constructed from F_i by applying CONTRACTION($2k$) ($i = 1, \dots, t-1$). Thus $n(H)$ is even. Let z_1, z_2, \dots, z_{2d} be the set of the vertices of H ordered in such a way that z_{2i-1}, z_{2i} correspond to the classes of the same partition $\tau_i = (Y_{2i-1}, Y_{2i})$ occurring in the course of applying CONTRACTION($2k$) ($i = 1, \dots, d$).

Let us denote by Z_j the set of vertices of F_1 contracted into z_j ($j = 1, \dots, 2d$). Because of the induction assumption the partition $\sigma_i = (Z_{2i-1}, Z_{2i})$ of the subgraph T_i induced on $Z_{2i-1} \cup Z_{2i}$ is admissible ($i = 1, \dots, d$). Therefore, if $m = |Z_j|$ ($j = 1, \dots, 2d$), $p_i = p(T_i)$, and $c_i = c_{\sigma_i}(T_i)$ then

$$c_i \leq \frac{1}{2} p_i \left(1 + \frac{1}{2m-1} \right) = \frac{p_i \cdot m}{2m-1} \quad (i = 1, \dots, d).$$

Let us assume for simplicity that d is even. (The odd d -case is similar). If $p_{ij} = p(Z_{2i-1} \cup Z_{2i}, Z_{2j-1} \cup Z_{2j})$ ($i \neq j$) then every partition of F_1 , that does not cut $Z_{2i-1} \cup Z_{2i}$ for all $i = 1, \dots, d$ can be interpreted as a partition of H . If one of these partitions is admissible, an admissible partition of \bar{H}_1 will be returned. Therefore, assuming that none of such partitions is admissible we come to the following inequality.

$$\frac{1}{2} \left(\sum_{\substack{i,j=1 \\ i \neq j}}^d p_{i,j} \right) \left(1 + \frac{1}{d-1} \right) > \frac{1}{2} p \left(1 + \frac{1}{2dm-1} \right),$$

or

$$\frac{2 \cdot m \cdot p \cdot (d-1)}{2dm-1} < \bar{p} \quad (*)$$

where $\bar{p} = \sum_{\substack{i,j=1 \\ i \neq j}}^d p_{i,j}$.

Let us consider partitions of F_1 that cut every set $Z_{2i-1} \cup Z_{2i}$ into the parts Z_{2i-1} and Z_{2i} ($i = 1, \dots, d$). It is clear that

- every such partition can be originated from some partition of H ,
- the minimal partition of such a kind cuts not more than $\frac{1}{2} \bar{p} + \sum_{i=1}^d c_i$ edges. So,

$$\begin{aligned}
 c_{\tau}(F_1) &\leq \frac{1}{2}\bar{p} + \sum_{i=1}^d c_i \leq \frac{1}{2}\bar{p} + \frac{m}{2m-1} \sum_{i=1}^d p_i \\
 &= \frac{1}{2}\bar{p} + \frac{m}{2m-1}(p - \bar{p}) = \frac{m}{2m-1}p - \bar{p} \frac{1}{2(2m-1)}.
 \end{aligned}$$

Using (*) we come to

$$\begin{aligned}
 c_{\tau}(F_1) &< \frac{m}{2m-1}p - \frac{m(d-1)}{(2dm-1)(2m-1)} \cdot p = \frac{m}{2m-1}p \left(1 - \frac{(d-1)}{2dm-1}\right) \\
 &= \frac{mp(2dm-d)}{(2m-1)(2dm-1)} = \frac{mdp}{2dm-1}.
 \end{aligned}$$

The proof is complete.

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