On degree-colorings of multigraphs

Mark K. Goldberg
Department of Computer Science,
Rensselaer Polytechnic Institute
Troy, NY, 12180.
goldbm4@rpi.edu

December 21, 2016

Abstract

A notion of degree-coloring is introduced; it captures some, but not all properties of standard edge-coloring. We conjecture that the smallest number of colors needed for degree-coloring of a multigraph \( G \) [the degree-coloring index \( \tau(G) \)] equals \( \max\{\Delta, \omega\} \), where \( \Delta \) and \( \omega \) are the maximum vertex degree in \( G \) and the multigraph density, respectively. We prove that the conjecture holds iff \( \tau(G) \) is a monotone function on the set of multigraphs.

1 Introduction.

The chromatic index \( \chi'(G) \) of a multigraph \( G(V, E) \) is the minimal number of colors (positive integers) that can be assigned to the edges of \( G \) so that no two adjacent edges receive the same color. Clearly, \( \Delta(G) \leq \chi'(G) \), where \( \Delta(G) \) is the maximal vertex degree in \( G \). The famous result by Vizing ([10]) establishes \( \chi' = \chi'(G) \leq \Delta(G) + p(G) \), where \( p(G) \) is the maximal number of parallel edges in \( G \). For graphs, in particular, \( \Delta(G) \leq \chi'(G) \leq \Delta(G) + 1 \). The problem of computing the exact value of the chromatic index was proved by Holyer ([5]) to be NP-hard even for cubic graphs. It is suspected that for every multigraph with \( \chi'(G) > \Delta(G) + 1 \), its chromatic index is determined by the parameter \( \omega(G) \), called the multigraph density:

\[
\omega(G) = \max_{H \subseteq G} \left\lfloor \frac{e(H)}{v(H)/2} \right\rfloor,
\]

where \( H \) is a sub-multigraph of \( G \), and \( v(H) \) (resp. \( e(H) \)) denotes the number of vertices (resp. edges) in \( H \). It is easy to prove that \( \omega(G) \leq \chi'(G) \) for every multigraph \( G \). Seymour in [7] and Stahl in [8] proved the equality \( \max(\Delta(G), \omega(G)) = \chi'^*(G) \), where \( \chi'^*(G) \) is the fractional chromatic index of \( G \), known to be polynomially computable (see [6]).

The following variation of the multigraph density idea was considered in [3]. Let \( \pi(F) \) denote the size of a maximum matching composed of the edges in a set \( F \subseteq E \). Denote
\[ \omega^*(G) = \max_{F \subseteq E} \left\lceil \frac{|F|}{\pi(F)} \right\rceil. \] Then, it is easy to see that
\[ \omega(G) \leq \omega^*(G) \leq \chi'(G). \]

It turns out (see [3]) that\[ \omega^*(G) = \max(\Delta(G), \omega(G)). \]

Conjectures connecting \[ \chi'(G), \omega(G), \text{and } \Delta(G) \] were independently proposed by Goldberg ([1]) and Seymour ([7]) more than 30 years ago ([4],[9]). Currently, the strongest variation of the conjecture ([2]) is as follows:

**Conjecture 1** If \[ \Delta(G) \neq \omega(G), \text{then } \chi'(G) = \max(\Delta(G), \omega(G)), \text{else } \chi'(G) \leq \Delta(G) + 1. \]

Every edge-coloring with colors 1, 2, \ldots, c yields an assignment \( \mu : V \rightarrow 2^{[1,c]} \), where for every \( x \in V \), \( \mu(x) \) denotes the set of colors used on the edges incident to \( x \). Given \( S \subseteq V \) and \( i \in [1,c] \), the set of vertices \( x \in S \) such that \( i \in \mu(x) \) is denoted \( S(i) \). It is easy to prove that the assignment \( \mu \) originated by an edge-coloring using colors 1, \ldots, c satisfies the following three conditions:

**Degree condition:** \[ \forall x \in V(G), |\mu(x)| = \deg_G(x); \]

**Cover condition:** \[ \forall S \subseteq V, |E(S)| \leq \sum_{i=1}^c \left\lfloor \frac{|S(i)|}{2} \right\rfloor; \]

**Matching condition:** \[ \forall i \in [1,c], \text{the submultigraph induced on } V^{(i)} \text{ either has a perfect matching, or is empty.} \]

**Definition 1** An assignment \( \mu : V(G) \rightarrow 2^{[1,c]} \) satisfying the degree and the cover conditions is called a degree-coloring.

Straightforward checking of the assignment presented in the Figure below shows that the assignment is a degree-coloring of the multicycle \( C \). However, it is not originated by any edge-coloring of \( C \), since the submultigraph of \( C \) induced on \( V^{(6)} \) has no perfect matching.

Let \( \tau(G) \) denote the smallest integer \( c \) for which a degree-coloring of \( G \) exists. It is easy to prove

**Lemma 1** \[ \max(\Delta(G), \omega(G)) \leq \tau(G) \leq \chi'(G). \]

**Conjecture 2** (the \( \tau \)-conjecture): For every multigraph \( G \), \[ \tau(G) = \max(\Delta(G), \omega(G)). \]
A real-valued function \( \kappa(G) \) defined on the set of multigraphs is called **monotone** if for any multigraph \( G \) and any submultigraph \( H \subseteq G, \kappa(H) \leq \kappa(G) \). Clearly, \( \Delta(G) \) and \( \omega(G) \) are monotone functions.

**Conjecture 3** The degree-coloring index \( \tau(G) \) is a monotone function on multigraphs.

It is easy to see that Conjecture 2 implies Conjecture 3. We prove in this paper that the reverse is also true: the monotonicity of \( \tau(G) \) implies Conjecture 2.

We use the standard graph-theoretical terminology which can be found in [11].

## 2 Monotonicity of \( \tau(G) \) and the \( \tau \)-conjecture.

It is easy to construct a \( \tau(G) \)-degree-coloring for a regular multigraph \( G \) with \( \omega(G) \leq \Delta(G) \).

**Lemma 2** If \( G \) is a \( \Delta \)-regular multigraph, and \( \omega(G) \leq \Delta \), then \( \tau(G) = \Delta \).

**Proof.** From the definition, \( \tau(G) \geq \Delta \). Consider the following assignment:

\[
\forall x \in V(G), \quad \mu(x) = \{1, 2, \ldots, \Delta\}.
\]

Given \( S \subseteq V(G), \forall i \in [1, \Delta], S_i(\mu) = S \). Thus,

\[
\sum_{i=1}^{\Delta} \left\lfloor \frac{|S_i(\mu)|}{2} \right\rfloor = \left\lfloor \frac{|S|}{2} \right\rfloor \Delta.
\]

Since \( \omega(G) \leq \Delta \), for any \( S \subseteq V, \left\lfloor \frac{|S|}{2} \right\rfloor \Delta \geq \left\lfloor \frac{|S|}{2} \right\rfloor \omega(G) \geq |E(S)| \) implying \( \tau(G) = \Delta \). \( \square \)

Constructing a degree-coloring for a non-regular multigraph can be done via operation Regularization which, for every multigraph \( G \), creates a regular multigraph \( R(G) \) containing \( G \) as an induced sub-multigraph.

**Regularization:** If a multigraph \( G \) is regular and \( \omega(G) \leq \Delta(G) \), then \( R(G) = G \); else

1. generate a disjoint isomorphic copy \( G' = (V', E') \) of \( G(V, E) \) with an isomorphic mapping \( f : V \rightarrow V' \) from \( G \) onto \( G' \);
2. let \( V(R(G)) = V \cup V' \) and initialize \( E(R(G)) \) by setting \( E(R(G)) = E(G) \cup E(G') \);
3. \( \forall x \in V, \) add \( \max(\Delta(G), \omega(G)) - \deg_G(x) \) new edges \( xf(x) \) to \( E(R(G)) \).

**Lemma 3** \( \forall G, \omega(G) \leq \omega(R(G)) \leq \max(\omega(G), \Delta(G)) \) and \( \Delta(R(G)) = \max(\Delta(G), \omega(G)) \).
Proof. If $G = R(G)$, the lemma is obvious. Let $G \neq R(G)$. Denote $\Delta = \Delta(G)$, $\omega = \omega(G)$, and $\rho = \max(\Delta, \omega)$. Obviously, $\Delta(R(G)) = \rho$ and $\omega \leq \omega(R(G))$.

To prove $\omega(R(G)) \leq \rho$, denote $R = R(G)$, $V(R) = V_1 \cup V_2$, where $V_1 = V(G)$ and $V_2 = V(G')$. Let $f$ be an isomorphic mapping from $V_1$ onto $V_2$. Given $S \subseteq V(R)$, let $S_1 = S \cap V_1$, $S_2 = S \cap V_2$, $S' = S_1 \cap f^{-1}(S_2)$, and $S'' = S_2 \cap f(S_1)$. Note that $|S'| = |S''|$ and $|E(S')| = |E(S'')|.

\[
\begin{array}{c}
\text{V}_1 \\
\hdashline
S_1 \\
S' \\
\hdashline
S_2 \\
\text{V}_2
\end{array}
\]

Then $|E(S)| = |E(S_1)| + |E(S_2)| + \sum_{x \in S'} (\rho - \deg_G(x))$

\[
= |E(S_1 - S')| + |E(S_1 - S', S')| + |E(S')| + |E(S_2 - S'')| + |E(S_2 - S', S'')| + |E(S'')| + |S'| \rho - \sum_{x \in S'} \deg_G(x).
\]

It is easy to check that

\[
|E(S_1 - S', S')| + |E(S')| + |E(S_2 - S'', S'')| + |E(S'')| = |E(S_1 - S', S')| + |E(S_2 - S'', S'')| + 2|E(S')| \leq \sum_{x \in S'} \deg_G(x),
\]

which yields the following upper bound

\[
|E(S)| \leq |E(S_1 - S')| + |E(S_2 - S'')| + |S'| \rho \leq \left[\frac{|S_1| - |S'|}{2}\right] \rho + \left[\frac{|S_2| - |S''|}{2}\right] \rho + |S'| \rho.
\]

To prove

\[
\left[\frac{|S_1| - |S'|}{2}\right] \rho + \left[\frac{|S_2| - |S''|}{2}\right] \rho + |S'| \rho \leq \left[\frac{|S_1| + |S_2|}{2}\right] \rho,
\]

note that it is straightforward if $|S_1| + |S_2|$ is even. If $|S_1| + |S_2|$ is odd, one out of two integers $|S_1| - |S'|$ and $|S_2| - |S''|$ is even and one is odd. Thus,

\[
\left[\frac{|S_1| - |S'|}{2}\right] \rho + \left[\frac{|S_2| - |S''|}{2}\right] \rho + |S'| \rho = \frac{|S_1|}{2} \rho + \frac{|S_2|}{2} \rho - \frac{1}{2} \rho.
\]
Since $|S_1| + |S_2|$ is odd,

$$\left\lfloor \frac{|S_1| + |S_2|}{2} \right\rfloor \rho = \frac{|S_1| + |S_2|}{2} \rho - \frac{1}{2} \rho,$$

which implies the result. \[ \square \]

**Theorem 1** If function $\tau(G)$ is monotone on the set of all multigraphs, then for any multigraph $G$,

$$\tau(G) = \max\{\Delta(G), \omega(G)\}.$$

**Proof.** By Lemma 1, $\max\{\Delta(G), \omega(G)\} \leq \tau(G)$. On the other hand, since $G \subseteq R(G)$, it follows from Lemma 2 that $\tau(G) \leq \tau(R(G)) = \max\{\Delta(G), \omega(G)\}$. \[ \square \]

**References**


