On degree-colorings of multigraphs

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December 21, 2016

Abstract

A notion of degree-coloring is introduced; it captures some, but not all properties of standard edge-coloring. We conjecture that the smallest number of colors needed for degree-coloring of a multigraph G [the degree-coloring index $\tau(G)$] equals $\max\{\Delta, \omega\}$, where Δ and ω are the maximum vertex degree in G and the multigraph density, respectively. We prove that the conjecture holds iff $\tau(G)$ is a monotone function on the set of multigraphs.

1 Introduction.

The **chromatic index** $\chi'(G)$ of a multigraph $G(V, E)$ is the minimal number of colors (positive integers) that can be assigned to the edges of G so that no two adjacent edges receive the same color. Clearly, $\Delta(G) \leq \chi'(G)$, where $\Delta(G)$ is the maximal vertex degree in G. The famous result by Vizing ([10]) establishes $\chi' = \chi'(G) \leq \Delta(G) + p(G)$, where $p(G)$ is the maximal number of parallel edges in G. For graphs, in particular, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. The problem of computing the exact value of the chromatic index was proved by Holyer ([5]) to be NP-hard even for cubic graphs. It is suspected that for every multigraph with $\chi'(G) > \Delta(G) + 1$, its chromatic index is determined by the parameter $\omega(G)$, called the multigraph **density**:

$$\omega(G) = \max_{H \subseteq G} \lceil \frac{e(H)}{\lfloor v(H)/2 \rfloor} \rceil,$$

where H is a sub-multigraph of G, and $v(H)$ (resp. $e(H)$) denotes the number of vertices (resp. edges) in H. It is easy to prove that $\omega(G) \leq \chi'(G)$ for every multigraph G. Seymour in [7] and Stahl in [8] proved the equality $\max(\Delta(G), \omega(G)) = \chi'^{*}(G)$, where $\chi'^{*}(G)$ is the the fractional chromatic index of G, known to be polynomially computable (see [6]).

The following variation of the multigraph density idea was considered in [3]. Let $\pi(F)$ denote the size of a maximum matching composed of the edges in a set $F \subseteq E$. Denote

$\omega^*(G) = \max_{F \subseteq E} \lceil \frac{|F|}{\pi(F)} \rceil$. Then, it is easy to see that

$$\omega(G) \leq \omega^*(G) \leq \chi'(G).$$

It turns out (see [3]) that $\omega^*(G) = \max(\Delta(G), \omega(G))$.

Conjectures connecting $\chi'(G), \omega(G)$, and $\Delta(G)$ were independently proposed by Goldberg ([1]) and Seymour ([7]) more than 30 years ago ([4],[9]). ([9]). Currently, the strongest variation of the conjecture ([2]) is as follows:

Conjecture 1 *If* $\Delta(G) \neq \omega(G)$, *then* $\chi'(G) = \max(\Delta(G), \omega(G))$, *else* $\chi'(G) \leq \Delta(G) + 1$.

Every edge-coloring with colors $1, 2, \ldots, c$ yields an assignment $\mu : V \to 2^{[1,c]}$, where for every $x \in V$, $\mu(x)$ denotes the set of colors used on the edges incident to x. Given $S \subseteq V$ and $i \in [1, c]$, the set of vertices $x \in S$ such that $i \in \mu(x)$ is denoted $S^{(i)}(\mu)$. It is easy to prove that the assignment μ originated by an edge-coloring using colors $1, \ldots, c$ satisfies the following three conditions:

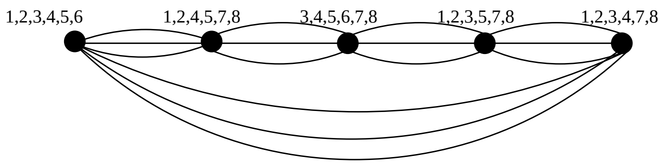
Degree condition: $\forall x \in V(G)$, $|\mu(x)| = deg_G(x)$;

Cover condition: $\forall S \subseteq V$, $|E(S)| \leq \sum_{i=1}^{c} \lfloor \frac{|S^{(i)}(\mu)|}{2} \rfloor$;

Matching condition: $\forall i \in [1, c]$, the submultigraph induced on $V^{(i)}$ either has a perfect matching, or is empty.

Definition 1 *An assignment* $\mu : V(G) \to 2^{[1,c]}$ *satisfying the degree and the cover conditions is called a* **degree-coloring**.

Straightforward checking of the assignment presented in the Figure below shows that the assignment is a degree-coloring of the multicycle C. However, it is not originated by any edge-coloring of C, since the submultigraph of C induced on $V^{(6)}$ has no perfect matching.



Let $\tau(G)$ denote the smallest integer c for which a degree-coloring of G exists. It is easy to prove

Lemma 1 $\max(\Delta(G), \omega(G)) \leq \tau(G) \leq \chi'(G)$.

Conjecture 2 *(the* τ*-conjecture): For every multigraph* G, $\tau(G) = \max(\Delta(G), \omega(G))$.

A real-valued function $\kappa(G)$ defined on the set of multigraphs is called **monotone** if for any multigraph G and any submultigraph $H \subseteq G$, $\kappa(H) \leq \kappa(G)$. Clearly, $\Delta(G)$ and $\omega(G)$ are monotone functions.

Conjecture 3 *The degree-coloring index $\tau(G)$ is a monotone function on multigraphs.*

It is easy to see that Conjecture 2 implies Conjecture 3. We prove in this paper that the reverse is also true: the monotonicity of $\tau(G)$ implies conjecture 2.

We use the standard graph-theoretical terminology which can be found in [11].

2 Monotonicity of $\tau(G)$ and the τ-conjecture.

It is easy to construct a $\tau(G)$-degree-coloring for a regular multigraph G with $\omega(G) \leq \Delta(G)$.

Lemma 2 *If G is a Δ-regular multigraph, and $\omega(G) \leq \Delta$, then $\tau(G) = \Delta$.*

Proof. From the definition, $\tau(G) \geq \Delta$. Consider the following assignment:

$$\forall x \in V(G), \ \mu(x) = \{1, 2, \ldots, \Delta\}.$$

Given $S \subseteq V(G)$, $\forall i \in [1, \Delta]$, $S^{(i)}(\mu) = S$. Thus,

$$\sum_{i=1}^{\Delta} \lfloor \frac{|S^{(i)}(\mu)|}{2} \rfloor = \lfloor \frac{|S|}{2} \rfloor \Delta.$$

Since $\omega(G) \leq \Delta$, for any $S \subseteq V$, $\lfloor \frac{|S|}{2} \rfloor \Delta \geq \lfloor \frac{|S|}{2} \rfloor \omega(G) \geq |E(S)|$ implying $\tau(G) = \Delta$. ∎

Constructing a degree-coloring for a non-regular multigraph can be done via operation Regularization which, for every multigraph G, creates a regular multigraph $R(G)$ containing G as an induced sub-multigraph.

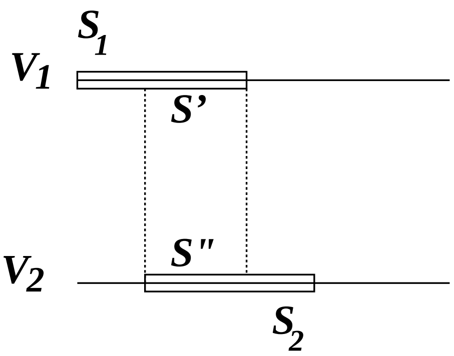
Regularization: If a multigraph G is regular and $\omega(G) \leq \Delta(G)$, then $R(G) = G$; else

1. generate a disjoint isomorphic copy $G' = (V', E')$ of $G(V, E)$ with an isomorphic mapping $f : V \to V'$ from G onto G';
2. let $V(R(G)) = V \cup V'$ and initialize $E(R(G))$ by setting $E(R(G)) = E(G) \cup E(G')$;
3. $\forall x \in V$, add $\max(\Delta(G), \omega(G)) - deg_G(x)$ new edges $xf(x)$ to $E(R(G))$.

Lemma 3 *$\forall G$, $\omega(G) \leq \omega(R(G)) \leq \max(\omega(G), \Delta(G))$ and $\Delta(R(G)) = \max(\Delta(G), \omega(G))$.*

Proof. If $G = R(G)$, the lemma is obvious. Let $G \neq R(G)$. Denote $\Delta = \Delta(G)$, $\omega = \omega(G)$, and $\rho = \max(\Delta, \omega)$. Obviously, $\Delta(R(G)) = \rho$ and $\omega \leq \omega(R(G))$.

To prove $\omega(R(G)) \leq \rho$, denote $R = R(G)$, $V(R) = V_1 \cup V_2$, where $V_1 = V(G)$ and $V_2 = V(G')$. Let f be an isomorphic mapping from V_1 onto V_2. Given $S \subseteq V(R)$, let $S_1 = S \cap V_1$, $S_2 = S \cap V_2$, $S' = S_1 \cap f^{-1}(S_2)$, and $S'' = S_2 \cap f(S_1)$. Note that $|S'| = |S''|$ and $|E(S')| = |E(S'')|$.



$$\begin{aligned}
\text{Then } |E(S)| &= |E(S_1)| + |E(S_2)| + \sum_{x \in S'} (\rho - deg_G(x)) \\
&= |E(S_1 - S')| + |E(S_1 - S', S')| + |E(S')| + \\
&\quad |E(S_2 - S'')| + |E(S_2 - S'', S'')| + |E(S'')| + |S'|\rho - \sum_{x \in S'} deg_G(x).
\end{aligned}$$

It is easy to check that

$$|E(S_1 - S', S')| + |E(S')| + |E(S_2 - S'', S'')| + |E(S'')| =$$
$$|E(S_1 - S', S')| + |E(S_2 - S'', S'')| + 2|E(S')| \leq \sum_{x \in S'} deg_G(x)),$$

which yields the following upper bound

$$\begin{aligned}
|E(S)| &\leq |E(S_1 - S')| + |E(S_2 - S'')| + |S'|\rho \\
&\leq \lfloor \frac{|S_1| - |S'|}{2} \rfloor \rho + \lfloor \frac{|S_2| - |S''|}{2} \rfloor \rho + |S'|\rho.
\end{aligned}$$

To prove

$$\lfloor \frac{|S_1| - |S'|}{2} \rfloor \rho + \lfloor \frac{|S_2| - |S''|}{2} \rfloor \rho + |S'|\rho \leq \lfloor \frac{|S_1| + |S_2|}{2} \rfloor \rho,$$

note that it is straightforward if $|S_1| + |S_2|$ is even. If $|S_1| + |S_2|$ is odd, one out of two integers $|S_1| - |S'|$ and $|S_2| - |S''|$ is even and one is odd. Thus,

$$\lfloor \frac{|S_1| - |S'|}{2} \rfloor \rho + \lfloor \frac{|S_2| - |S''|}{2} \rfloor \rho + |S'|\rho = \frac{|S_1|}{2}\rho + \frac{|S_2|}{2}\rho - \frac{1}{2}\rho.$$

Since $|S_1| + |S_2|$ is odd,

$$\lfloor \frac{|S_1| + |S_2|}{2} \rfloor \rho = \frac{|S_1| + |S_2|}{2}\rho - \frac{1}{2}\rho,$$

which implies the result. ∎

Theorem 1 *If function $\tau(G)$ is monotone on the set of all multigraphs, then for any multigraph G,*

$$\tau(G) = \max\{\Delta(G), \omega(G)\}.$$

Proof. By Lemma 1, $\max\{\Delta(G), \omega(G)\} \leq \tau(G)$. On the other hand, since $G \subseteq R(G)$, it follows from Lemma 2 that $\tau(G) \leq \tau(R(G)) = \max\{\Delta(G), \omega(G)\}$. ∎

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