Homework Problems

1. Problem 3.25(c)-(d) [10 POINTS]:
   (c) To formulate the statement “Nobody is bald” using only predicates \( P(x, h) = \text{“Person } x \text{ has hair } h \” \) and \( M(h) = \text{“Hair } h \text{ is grey,”} \) a possible solution is:
   \[
   \forall x : (\exists h : P(x, h))
   \]
   Note that given the ambiguity of the English language, multiple interpretations are possible here.
   (d) To formulate the statement “Kilam does not have all grey hair” using only predicates
   \( P(x, h) = \text{“Person } x \text{ has hair } h \” \) and \( M(h) = \text{“Hair } h \text{ is grey,”} \) a possible solution is:
   \[
   \exists h : P(\text{Kilam}, h) \land \neg M(h))
   \]
   Other solutions may exist, so please review each for correctness.

2. Problem 3.29(c)-(d) [10 POINTS]:
   (c) Given the predicates \( F(x) = \text{“} x \text{ is a freshman} \) and \( M(x) = \text{“} x \text{ is a math major} \),
   the statement \( \forall x : (M(x) \rightarrow \neg F(x)) \) translates to “If someone is a math major, then this person is not a freshman.”
   (d) Given the predicates \( F(x) = \text{“} x \text{ is a freshman} \) and \( M(x) = \text{“} x \text{ is a math major} \),
   the statement \( \neg \exists x : (M(x) \land \neg F(x)) \) translates to “Nobody is both a math major and NOT a freshman.”

3. Problem 3.36(c)-(d) [20 POINTS]: First note that \( N \subseteq Z \subseteq Q \subseteq R \).
   (c) The claim \( \forall x : (\exists y : x^2 = y) \) is \( T \) for the following domain pairs (and the implied larger sets, e.g., \( x \in \mathbb{N} \) and \( y \in \mathbb{Z} \), etc.):
      (i) \( x, y \in \mathbb{N}^2 \)
      (ii) \( x \in \mathbb{Z} \) and \( y \in \mathbb{Z} \) (or \( y \in \mathbb{N}_0 \))
      (iii) \( x \in \mathbb{Q} \) and \( y \in \mathbb{Q} \)
      (iv) \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \)
   (d) The claim \( \forall y : (\exists x : x^2 = y) \) is \( T \) only for \( y \in \mathbb{N} \) and \( x \in \mathbb{R} \).
4. **Problem 4.8(c)-(d) [10 POINTS]:**

(c) **Proof.** We prove the claim \(2^n - 1 \text{ is prime} \rightarrow n \text{ is prime} \) using contraposition.

1. Assume that \(q\) is \(F\), i.e., that \(n\) is not prime.
2. This means that \(n = ab \) with \(a, b \in \mathbb{N}, a > 1, \) and \(b > 1\).
3. Observe that \(2^n - 1 = 2^{ab} - 1\). Let \(m = 2^a\) and note that since \(a > 1,\) we have \(m > 2\) (or \(m > 3\)).
4. Then, \(2^n - 1 = m^b - 1 = (m - 1) \times (\sum_{i=0}^{b-1} m^i)\).
5. Therefore, \(m^b - 1\) is divisible by first term \((m - 1)\) for \(m - 1 > 1\).
6. Since \(m^b - 1\) is shown to not be prime, \(2^n - 1\) is not prime.
7. Therefore, the statement claimed in \(p\) is \(F\).

(d) **Proof.** We prove the claim \((n^3 \text{ is odd} \rightarrow n \text{ is odd})\) using contraposition.

1. Assume that \(q\) is \(F\), i.e., that \(n\) is even.
2. This means that \(n = 2k\) for \(k \in \mathbb{Z}\).
3. Observe that \(n^3 = (2k)^3 = 8k^3\), indicating that \(n^3\) is a multiple of 8.
4. Since \(n^3\) is a multiple of 8, \(n^3\) is even.
5. Therefore, the statement claimed in \(p\) is \(F\).

5. **Problem 4.10(f) and 4.10(h) [20 POINTS]:**

(f) **Proof.** We prove the claim \(((x, y) \in \mathbb{Z}^2 \rightarrow x^2 - 4y - 3 \neq 0)\) using contradiction.

Assume \(x^2 - 4y - 3 = 0\), then rewrite as \(x^2 = 4y + 3\).

Next, consider the cases in which \(x\) is even or odd.

If \(x\) is even, then \(x = 2k\) \((k \in \mathbb{Z})\), and we have \(x^2 = (2k)^2 = 4k^2\).

If instead \(x\) is odd, then \(x = 2k + 1\) \((k \in \mathbb{Z})\), and we have \(x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1\).

From both of these cases, we observe that \(x^2\) is either a multiple of 4 or one more than a multiple of 4.

Since the RHS \(4y + 3\) is three more than a multiple of 4, we have a contradiction.

Therefore, we conclude that \((x, y) \in \mathbb{Z}^2 \rightarrow x^2 - 4y - 3 \neq 0\).

(h) **Proof.** We prove the claim \(\forall(a, b, c) \in \mathbb{Z}^3 : (a^2 + b^2 = c^2) \rightarrow (a \text{ or } b \text{ is even})\) using contradiction.

Assume neither \(a\) nor \(b\) is even, i.e., \(a\) and \(b\) are both odd.

Then \(a = 2k + 1\) and \(b = 2m + 1\) \((k, m \in \mathbb{Z})\).

Plugging this in, we have \((2k + 1)^2 + (2m + 1)^2 = c^2\), which we rewrite as \(4k^2 + 4k + 1 + 4m^2 + 4m + 1 = c^2\) or \(4(k^2 + k + m^2 + m) + 2 = c^2\).

Next, consider the cases in which \(c\) is even or odd.

If \(c\) is odd, then \(c^2\) is also odd, but the LHS above must be even, so we have a contradiction.

If \(c\) is even, then it is divisible by 2 and \(c^2\) is divisible by 4, but the LHS above is not divisible by 4, so we have a contradiction.

Therefore, using either contradiction above, we conclude that \(\forall(a, b, c) \in \mathbb{Z}^3 : (a^2 + b^2 = c^2) \rightarrow (a \text{ or } b \text{ is even})\).
6. Problem 4.14(o) [10 POINTS]:

(o) Proof. We disprove the claim (\(\exists x, y \in \mathbb{Z}^2\) for which \(2x^2 + 5y^2 = 14\)) by using one of the two approaches below.

(i) Both terms \(x^2\) and \(y^2\) are non-negative, so we can determine the maximum for each that might get us to 14. For \(2x^2\), we have \(x = 3\) as a maximum (with \(2(3^2) = 18\)), and for \(5y^2\), we have \(y = 2\) as a maximum (with \(5(2^2) = 20\)). Finally, pick either \(x\) or \(y\) and go through each possible value to show we can never obtain the sum of 14.

(ii) Use contradiction. Assume that we have pair \(x, y\) for which the equation holds. Rewrite \(2x^2 + 5y^2 = 14\) to \(5y^2 = 14 - 2x^2\). Since both 14 and \(2x^2\) are even, the LHS \(5y^2\) must also be even, indicating that \(y\) must be even. If \(y\) is even, then \(y = 2k\) with \(k \in \mathbb{Z}\). This implies that \(2x^2 + 5(2k)^2 = 14\) or \(2x^2 + 20k^2 = 14\), which simplifies to \(x^2 + 10k^2 = 7\). We observe that it must be the case that \(k = 0\) here, leading to \(x^2 = 7\), which is impossible, so we have our contradiction.

7. Problem 4.26(c) [10 POINTS]:

(c) Proof. See similar problem and proof in the Chapter 4 lecture notes (from 1/29).

8. Problem 4.27(c) [10 POINTS]:

(c) Proof. We show that \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\) using a formal proof. The formal proof has two main steps, one for each implication:

(i) We show \(A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)\) via a direct proof.
   Assume \(x \in A \cap (B \cup C)\). This means that \(x \in A \cap B\) or \(x \in A \cap C\).
   Therefore, \(x \in (A \cap B) \cup (A \cap C)\).

(ii) Next, we show \((A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)\) via contraposition.
    Suppose that \(x \notin A \cap (B \cup C)\). This means that \(x \notin A \cap B\) and \(x \notin A \cap C\).
    Therefore, \(x \notin (A \cap B) \cup (A \cap C)\).

Finally, since we proved \(A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)\) and \((A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)\), we conclude that \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\).