Homework Problems

1. **Problem 4.13(h) and 4.13(l) [20 POINTS]:**

   Determine the type of proof and prove.

   (h) *Proof.* We prove the claim (if \( n \in \mathbb{Z} \), then \( n^2 - 3 \) is not divisible by 4) by using contradiction. Assume that \( q \) is \( F \), which indicates that \( n^2 - 3 \) is divisible by 4. If \( n^2 - 3 \) is divisible by 4, this implies that \( n^2 - 3 = 4k \), with \( k \in \mathbb{Z} \). Rewrite this as \( n^2 = 4k + 3 \) and consider the case in which \( n \) is even. If \( n \) is even, then \( n^2 \) is even, but \( 4k + 3 \) is odd, thus \( n^2 \neq 4k + 3 \), so we have found a contradiction. Similarly, if \( n \) is odd, then \( n^2 \) is odd, meaning that \( n = 2m + 1 \). Plugging that in, we have \((2m + 1)^2 = 4k + 3\), which we can rewrite as \( 4m^2 + 4m + 1 = 4k + 3 \) and rearrange as \( 4(m^2 + m) = 4k + 2 \) and then \( 2(m^2 + m) = 2k + 1 \). The LHS is even, but the RHS is odd, thus we have found another contradiction. We conclude that the original claim must therefore be \( T \).

   (l) *Proof.* We prove the claim (When dividing \( n \) by \( d \), the quotient \( q \) and remainder \( 0 \leq r < d \) are unique) by using contradiction. To do so, we can assume a form from the divisibility theorem and use it to prove the stronger claim “If we divide \( n \) by \( d \), the quotient \( q \) and remainder \( 0 \leq r < d \) must be unique.”

   Assume that \( q \) is \( F \) but that \( p \) still holds. If \( p \) is \( T \), then we can express it as \( n = dq + r \).

   Since we assume \( q \) is \( F \), we assume \( q \) and \( r \) are not unique, indicating that we can find two sets of integers \((q_1, r_1)\) and \((q_2, r_2)\) that satisfy the above equation, i.e., \( n = dq_1 + r_1 \) and \( n = dq_2 + r_2 \).

   For simplicity sake, assume that \( r_2 > r_1 \) (we could also assume the opposite); then we know that \( 0 \leq r_1 < r_2 < d \).

   Combining the above equations, we have:

   \[
   dq_2 + r_2 = dq_1 + r_1 \\
   d(q_2 - q_1) = (r_2 - r_1) \\
   (q_2 - q_1) = \frac{(r_2 - r_1)}{d}.
   \]

   Next, since \( r_2 > r_1 \), \( r_2 < d \), and \( r_1 \geq 0 \), we have \( 0 \leq r_2 - r_1 < d \).

   Dividing both sides by \( d \), we get:

   \[
   0 \leq \frac{r_2 - r_1}{d} < 1.
   \]

   But we also showed that \( (q_2 - q_1) = \frac{(r_2 - r_1)}{d} \).

   And \( (q_2 - q_1) \) must be an integer, as the difference of two integers is an integer.

   Since \( 0 \leq \frac{r_2 - r_1}{d} < 1 \), we observe that the only possible integer in this range is 0.

   Therefore, we can say that \( q_2 - q_1 = 0 \), meaning that \( q_2 = q_1 \) and similarly \( r_2 = r_1 \).

   We have then shown that \( q \) and \( r \) must be unique by contradiction. The original implication holds.
2. Problem 4.28(b)-(c) [14 POINTS]:
In each question, prove (or disprove) a relationship between the sets.

(b) Given \( A = \{7k, k \in \mathbb{N}\} \) and \( B = \{3k, k \in \mathbb{N}\} \). Prove that \( A \cap B \neq \emptyset \).

Proof. We prove the claim by providing an example. Here, \( 21 \in A \) and \( 21 \in B \). Therefore, \( A \cap B \neq \emptyset \).

Note this can also be proven by contradiction as follows. Assume that \( A \cap B = \emptyset \).

We expand sets \( A \) and \( B \) as \( A = \{7, 14, 21, 28, \ldots \} \) and \( B = \{3, 6, 9, 12, 15, 18, 21, 24, \ldots \} \).

We observe that \( 21 \in (A \cap B) \), which contradicts \( A \cap B \neq \emptyset \).
Therefore, we have proven the original claim.

(c) Given \( A = \{4k - 3, k \in \mathbb{N}\} \) and \( B = \{4k + 1, k \in \mathbb{N}\} \). Prove or disprove \( A = B \).

To prove that \( A = B \), we need to show that all elements of \( B \) are in \( A \) and that all elements of \( A \) are in \( B \).

Conversely, to disprove \( A = B \), we need only one example that shows an element in either \( A \) or \( B \) that is not in both sets.

We observe that \( A = \{1, 5, 9, 13, \ldots \} \) and \( B = \{5, 9, 12, 15, \ldots \} \).

More specifically, \( k = 1 \) yields the smallest possible value in each set \( A \) and \( B \).

Therefore, we conclude that \( 1 \notin B \) and therefore \( A \neq B \).

3. Problem 5.3(c) [6 POINTS]:
For which \( n \) is \( P(n) \) is \( T \)? Explain by showing the “chain” of implications.

(c) \( P(2) \) is \( T \) and \( P(n) \implies (P(n^2) \land P(n-2)) \) is \( T \) for \( n \geq 2 \).

Start with the solution for the base case in which \( n = 2 \).

When \( n = 2 \), \( P(2) \) is \( T \), which implies that \( P(2^2) \land P(2 - 2) \) is \( T \), which translates to \( P(4) \) and \( P(0) \) being \( T \).

For \( P(0) \), no further implications can be made, but \( P(4) \) implies \( P(16) \) and \( P(14) \).

Further, \( P(16) \) implies both \( P(256) \) and \( P(14) \); and \( P(14) \) implies both \( P(196) \) and \( P(12) \).

The pattern that emerges here is that \( P(n) \) is \( T \) for all even numbers \( n \geq 0 \).
4. **Problem 5.10(d) [10 POINTS]:**

For $n \geq 1$, prove by induction.

(d) **Proof.** For $n \geq 1$, we prove by induction that $P(n)$: $3^n > n^2$ is $T$.

For the base case, i.e., $n = 1$, we have $3^1 > 1^2$ or $3 > 1$, which is $T$.

Next, we assume that claim $P(n)$ is $T$, i.e., $3^n > n^2$ is our inductive hypothesis.

We need to show that $P(n + 1)$ is $T$, i.e., that $3^{n+1} > (n + 1)^2$.

We start with the LHS of $P(n + 1)$, rewriting it as $3^{n+1} = 3 \times 3^n$.

We then use the inductive hypothesis to write $3 \times 3^n > 3n^2$, but we need to show that $3^{n+1} > n^2 + 2n + 1$.

Going back to $3 \times 3^n > 3n^2$, we can rewrite the RHS as $3n^2 = n^2 + n^2 + n^2$.

Therefore, we can rewrite the above as $3^{n+1} > n^2 + n^2 + n^2$ and $3^{n+1} > n^2 + 2n + 1$.

Comparing the above two equations, we need to show that $2n^2 \geq 2n + 1$ for $n \geq 2$ (since we already confirmed $P(1)$ above).

We can use another inductive proof to prove this. For the base case, we use $n = 2$ and confirm that $2n^2 \geq 2n + 1$ or $8 \geq 5$.

For the inductive step, we assume that $P(n)$ is $T$, i.e., $2n^2 \geq 2n + 1$ is our inductive hypothesis.

We need to show that $2(n + 1)^2 \geq 2(n + 1) + 1$ or $(n + 1)^2 \geq 2n + 3$.

Starting with $2(n + 1)^2$, we rewrite this as $2n^2 + 4n + 2$.

Using the inductive hypothesis, we have $2n^2 + 4n + 2 \geq 2n + 1 + 4n + 2 = 6n + 3$. Next, we observe that $6n + 3 > 2n + 3$, thus we have proven that $2(n + 1)^2 > 2n + 3$.

We have then shown by induction that $2n^2 > 2n + 1$, and therefore we have shown by induction that $3^{n+1} > (n + 1)^2$.

5. **Problem 5.14 [10 POINTS]:**

Prove, by induction, that every $n \geq 1$ is a sum of distinct powers of 2.

**Proof.** We use induction to prove this claim.

For the base case of $n = 1$, we need to show that 1 can be expressed as a sum of distinct powers of 2. Here, we observe that $2^0 = 1$, hence we have proven the base case to be $T$.

For the induction step, we assume that $P(n)$ is $T$.

We then need to show that $P(n + 1)$ is $T$.

In the $n + 1$ case, we have that $P(n + 1) = n + 1$; in other words, we must add 1 to the given number. We consider the two cases below.

(i) If $n + 1$ is odd, then $n$ is even. To obtain $n + 1$ from $n$, we add $2^0$, which we know is not present for $n$ since $n$ is even (and all powers $2^k$ with $k > 1$ are even).

(ii) If $n + 1$ is even, then $n$ is odd. To obtain $n + 1$ from $n$, we use the well-ordering principle to note that there is a smallest value $m$ for which $2^m$ is not included in the sum of distinct power of 2 for $n$. We therefore add $2^m$ and remove all powers of 2 less than $2^m$ to obtain $n + 1$. 

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6. **Problem 5.16 [10 POINTS]**: 
Let \( A \) be a finite set of size \( n \geq 1 \); prove by induction that \(|\wp(A)| = 2^n\). **Proof.** We use induction to prove this claim. 
For the base case of \( n = 1 \), we must show that \(|\wp(A)| = 2\).
To do so, define set \( A \) that contains arbitrary element \( a \) (the domain of \( a \) is not important here). 
Then, \( A = \{a\} \) and we have power set \( \wp(A) = \{\emptyset, \{a\}\} \). Hence we have that \( |A| = 1 \) and \( |\wp(A)| = 2 \). Therefore, \( |\wp(A)| = 2^1 = 2 \).
For the induction step, we start by assuming that \( P(n) \) is \( T \). Hence we have that \(|\wp(A)| = 2^n\).
We need to show that \( P(n + 1) \) is \( T \), i.e., \(|\wp(A)| = 2^{n+1}\).
We know that \( |A|_{n+1} = |A|_n + 1 \). Therefore, we can build the set of \( n + 1 \) elements by adding an element to the set \( A \) of \( n \) elements.
When we build the power set of this augmented set \( A_{n+1} \), the newly added element contributes to every element in the power set; therefore, there are now \( 2 \times 2^n \) elements in the augmented set \( A_{n+1} \) power set.
Therefore, through induction, we have proven our initial claim.

7. **Problem 5.30 [10 POINTS]**: 
**Proof.** To prove that you can make any postage greater than 12¢ using only 4¢ and 5¢ stamps, we use leaping induction. 
We can formalize the claim as \( P(n) = 4m + 5n \), where \( m, n \in \mathbb{N}_0 \).
For the base case, we have \( P(13) = 4 + 4 + 5 \), \( P(14) = 4 + 5 + 5 \), \( P(15) = 5 + 5 + 5 \), and \( P(16) = 4 + 4 + 4 + 4 \). Therefore, \( P(13), P(14), P(15), \) and \( P(16) \) are \( T \).
For the induction step, we assume \( P(n) \) is \( T \). We must prove that \( P(n + 4) \) is \( T \). Observe that \( P(n + 4) = P(n) + 4 = 4m + 5n + 4 = 4(m + 1) + 5n \). Therefore, \( P(n + 4) \) is \( T \). And we conclude, by induction, that \( P(n) \) is \( T \) for all natural numbers \( n > 12 \).
8. Problem 5.47 [20 POINTS]:

(a) All possible moves \((\Delta x, \Delta y)\) are as follows: \((1, 1), (1, -1), (-1, 1),\) and \((-1, -1)\). Each move requires the robot to move two blocks, which could be described as \(|x| + |y| = 2\).

The square to right of the robot is an odd number of squares away, which means that no combination of its possible moves will lead the robot there.

(b) The first step is to show that all moves within the \(3 \times 3\) grid surrounding the robot are now possible. The list of all possible moves \((\Delta x, \Delta y)\) are as follows: \((1, 2), (1, -1), (-1, 1),\) and \((-1, -1)\).

We define \(\text{destination}(x, y) = a(1, 2) + b(1, -1) + c(-1, 1) + d(-1, -1)\), where \(a, b, c, d \in \mathbb{Z}^4\) represent the number of such moves in each direction.

Then from \((0, 0)\), we can move the robot to all eight surrounding adjacent squares as follows:

- top right corner: \((1, 1) = 2(1, 2) + 1(1, -1) + 2(-1, -1)\)
- top left corner: \((1, -1) = 1(1, -1)\)
- bottom left corner: \((-1, -1) = 1(-1, -1)\)
- bottom right corner: \((1, -1) = 1(1, -1)\)
- up: \((0, 1) = 1(1, 2) + 1(-1, -1)\)
- left: \((-1, 0) = 1(1, 2) + 2(-1, -1)\)
- down: \((1, 1) = 1(1, 2) + 2(-1, -1) + 1(1, -1)\)
- right: \((1, 1) = 1(1, 2) + 1(-1, -1) + 1(1, -1)\)

Given the above, for the base case, since all cases of \(|x| \leq 1\) and \(|y| \leq 1\) have been shown, we know that the robot can move to all eight surrounding adjacent squares.

For the induction step, we assume \(P(n)\) is \(T\), i.e., that the robot can move to any surrounding square \((\Delta x, \Delta y)\), where \(|x| \leq n\) and \(|y| \leq n\). Next, we must prove \(P(n + 1)\), i.e., that the robot can move to any square \((\Delta(x + 1), \Delta(y + 1))\), where \(|x + 1| \leq n + 1\) and \(|y + 1| \leq n + 1\).

Since we know that the robot can move to all eight surrounding adjacent squares, with \(|x| \leq 1\) and \(|y| \leq 1\), and the assumption is that the robot can move to any square \((x, y)\), with \(|x| \leq n\) and \(|y| \leq n\), we observe that \(|x + 1| \leq n + 1\) and \(|y + 1| \leq n + 1\) is a possible move.

Therefore, \(P(n + 1)\) is \(T\), and we conclude that any square \((m, n)\) can be reached by a finite sequence of moves.