Homework Problems

1. [10 POINTS] Problem 13.24(e)-(f) [or F17 Problem 13.17(e)-(f)]:
   In each case, count.
   (e) The number of orders in which a traveling salesman can visit the 50 states.
   **Solution:** In this problem, since order does matter, we are counting the number of $k$-orderings or $k$-permutations, where $k = 50$ and the number of elements we are ordering is $n = 50$. Therefore, we can calculate $q$ as the number of $k$-orderings as
   $$q = \frac{n!}{(n-k)!} = \frac{50!}{0!} = 50!$$

   This is a very large number! If you happen to have used a calculator for this, the answer is $3,041,409,320,171,337,804,361,260,816,6065 \times 10^{64}$ (yikes).

   (f) The number of poker hands with a card in every suit.
   **Solution:** A poker hand consists of exactly five cards. The problem requires four cards to be in the four different suits. This implies that two cards will be of the same suit; in this case, there are $\binom{13}{2}$ ways to select the two cards. For the other 3 suits, there are 13 ways to select each card. Finally, we multiply the $\binom{13}{2}$ by 4, one for each suit:
   $$q = 4 \times \binom{13}{2} \binom{13}{1} \binom{13}{1} = 4 \times \frac{13!}{2!(13-2)!} \times \frac{13!}{1!(13-1)!} = 685,464$$

2. [10 POINTS] Problem 13.38(b) [this problem is not in the F17 textbook]:
   What is the coefficient of $x^3$ in the expansion of:
   (b) $(3 - 2x)^6$
   **Solution:** We need to find the $(3)^i(-2x)^{6-i}$ term that produces the $x^3$ term.
   Here, $i = 3$.
   From the Binomial Theorem, the $x^3$ term is then
   $$\binom{6}{3}(3)^i(-2x)^{6-i} = \binom{6}{3}(3)^3(-2)^3x^3 = \frac{6!}{3!3!}(27)(-8)x^3 = -4320x^3$$

   The coefficient of $x^3$ in the expansion of $(3 - 2x)^6$ is $-4320$.

3. [8 POINTS] Problem 14.8 [or F17 Problem 14.5]:
   Sets $A$, $B$, $C$ have sizes 2, 3, 4. What are the min and max for $|A \cup B \cup C|$?
   **Solution:** $4 \leq |A \cup B \cup C| \leq 9$
4. **[8 POINTS] Problem 14.9(a) [this problem is not in the F17 textbook]:**

Consider the binary strings consisting of 10 bits.

(a) How many contain fewer 1’s than 0’s?

**Solution:** Overall, there are $2^{10} = 1024$ binary strings consisting of 10 bits. We must find $x$, the number of 10-bit binary strings containing fewer 1’s than 0’s.

There are at least two approaches here:

i. Observe that the answer will be symmetric with the number of 10-bit binary strings containing fewer 0’s than 1’s.

Therefore, first calculate the number of 10-bit binary strings with the same number of 0’s and 1’s as \( \binom{10}{5} = \frac{10!}{5!5!} = 252 \).

Then, based on symmetry, $x = \frac{1024 - 252}{2} = 386$.

ii. Observe that to have fewer 1’s than 0’s, a 10-bit binary string must contain $n$ 0’s, where $n$ takes on values \{6, 7, 8, 9, 10\}.

Therefore, we can calculate

$$x = \binom{10}{6} + \binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} = 210 + 120 + 45 + 10 + 1 = 386$$

5. **[10 POINTS] Problem 14.19(f)-(g) [or F17 Problem 14.11(f)-(g)]:**

In each case, count the number of objects/arrangements of the given type:

(f) US Social Security numbers (see Problem 13.22 [or F17 Problem 13.15]) with digits in strictly increasing order.

**Solution:** This problem is essentially asking you to count $x$ as the number of 9-digit sequences (i.e., leading zeroes are fine) in which each subsequent digit going from left to right increases. This is actually a rather small set of values:

\{012345678, 012345679, 012345689, 012345789, 012346789, 012356789, 012456789, 013456789, 023456789, 023456789, 023456789, 123456789\}

The number of sequences is therefore $x = 10$.

(g) US Social Security numbers with digits in non-decreasing order.

**Solution:** This problem is asking you to count $y$ as the number of 9-digit sequences (i.e., leading zeroes are fine) in which each subsequent digit going from left to right does not decrease. A good way to tackle this problem is to come up with a simpler representation. In this case, use delimiters between each digit, in order from 0 up to 9. Let each \( \diamond \) represent a digit, and each \( | \) represent a change from digit $n$ to digit $n + 1$.

There will always be exactly 9 \( \diamond \) characters (because we require a 9-digit sequence) and exactly 9 \( | \) delimiters, for an overall total of 18 characters.

As an example, \( \diamond\diamond\diamond||\diamond\diamond||\diamond\diamond|| \) represents the 9-digit sequence 000233677.

Given the above representation, we need to count the number of ways we can place 9 delimiters before/among/after 9 digits. To accomplish this, we need to choose 9 spots out of 18.

Therefore, $y = \binom{18}{9} = 48,620$. 

6. [12 POINTS] Problem 14.21 [or F17 Problem 14.12]: Use inclusion-exclusion to count the integer solutions to \( x_1 + x_2 + x_3 = 20 \) where \(-2 \leq x_1 \leq 10, 2 \leq x_2 \leq 8, \) and \( 0 \leq x_3 \leq 15. \)

**Solution:** Start by adjusting the ranges of \( x_1 \) and \( x_2 \) to be non-negative integers, so let \( y_1 = x_1 + 2 \) with \( 0 \leq y_1 \leq 12, y_2 = x_2 - 2 \) with \( 0 \leq y_2 \leq 6, \) and simply \( y_3 = x_3 \) with \( 0 \leq y_3 \leq 15. \)

From this, we use inclusion-exclusion to count the integer solutions of \( y_1 + y_2 + y_3 = 20. \)

This is a \( k \)-subset with repetition problem. From Chapter 13, we can make use of

\[
Q(n, k) = \binom{n + k - 1}{k - 1}
\]

More specifically, we start by counting the number of solutions with \( y_1, y_2, y_3 \geq 0; \) here, with \( n = 20 \) and \( k = 3, \) we have

\[
Q(n, k) = Q(20, 3) = \binom{22}{2} = 231
\]

Next (the exclusion part), let \( S_1 \) be the set of solutions that violate the upper bound constraint on \( y_1. \) Therefore, \( S_1 \) contains solutions to \( y_1 + y_2 + y_3 = 20 \) with \( y_1 \geq 13. \) Similarly, let \( S_2 \) be the set of solutions with \( y_2 \geq 7, \) and let \( S_3 \) be the set of solutions with \( y_3 \geq 16. \)

Given these sets, the number of solutions that satisfy \( y_1 + y_2 + y_3 = 20 \) is

\[
Q(20, 3) - |S_1 \cup S_2 \cup S_3|
\]

Using inclusion-exclusion, we have

\[
|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|
\]

To obtain \( |S_1|, \) observe that solutions to \( y_1 + y_2 + y_3 = 20 \) with \( y_1 \geq 13 \) are solutions to \( z_1 + y_2 + y_3 = 7, \) where \( z_1 = y_1 - 13 \) and \( z_1 \geq 0. \) Therefore, \( |S_1| = Q(7, 3) = \binom{9}{2} = 36. \) Using a similar approach, \( |S_2| = Q(13, 3) = \binom{15}{2} = 105 \) and \( |S_3| = Q(4, 3) = \binom{6}{2} = 15. \)

To determine \( |S_1 \cap S_2|, \) we observe that this intersection contains solutions with both \( y_1 \geq 13 \) and \( y_2 \geq 7. \) Therefore, \( |S_1 \cap S_2| = Q(0, 3) = \binom{2}{2} = 1. \) Similarly, \( |S_1 \cap S_3| = Q(-9, 3) = 0 \) and \( |S_2 \cap S_3| = Q(-3, 3) = 0. \)

Finally, the cardinality of the intersection of all three, i.e., \( |S_1 \cap S_2 \cap S_3| = Q(-16, 3) = 0. \)

Putting this all together yields

\[
|S_1 \cup S_2 \cup S_3| = 36 + 105 + 15 - 1 - 0 - 0 + 0 = 155
\]

Therefore, our answer is \( 231 - 155 = 76. \)
7. **[10 POINTS]** Problem 14.50(a)-(b) [or F17 Problem 14.34(a)-(b)]:

8 distinguishable dice are rolled. How many outcomes are there?

**Solution:** First, given 8 dice, there are $6^8 = 1,679,616$ possible outcomes.

(a) How many outcomes do not contain a 1? How many do not contain a 1 or 2?

**Solution:** To determine how many outcomes do not contain a 1, each die roll has 5 possible outcomes. Therefore, with 8 dice, we have $5^8 = 390,625$ possible outcomes that do not contain a 1.

Similarly, to determine how many outcomes do not contain a 1 or a 2, each die roll has 4 possible outcomes. Therefore, with 8 dice, we have $4^8 = 65,536$ possible outcomes that do not contain a 1 or a 2.

(b) How many outcomes contain all 6 numbers? *[Hint: Inclusion-exclusion.]*

**Solution:** As noted above, with 8 dice, there are $6^8 = 1,679,616$ possible outcomes.

Exclude from this the $5^8 = 390,625$ outcomes that do not contain a 1, the $5^8 = 390,625$ outcomes that do not contain a 2, etc. More specifically, we exclude $6 \times 5^8 = 2,343,750$ outcomes here.

Next, we include (i.e., add back) the $4^8 = 65,536$ outcomes that do not contain a 1 or a 2, since these were excluded twice in the previous step. To do this for each pair, we add back $\binom{6}{2} \times 4^8 = 983,040$.

Continuing this pattern, we have our answer $x$ calculated as

$$x = 6^8 - 6 \times 5^8 + \binom{6}{2} \times 4^8 - \binom{6}{3} \times 3^8 + \binom{6}{4} \times 2^8 - \binom{6}{5} \times 1^8$$

$$= 1,679,616 - 2,343,750 + 983,040 - 131,220 + 3840 - 6$$

$$= 191,520$$

8. **[12 POINTS]** Problem 15.7(a)-(c) [this problem is not in the F17 textbook]:

Randomly roll two dice. Compute the probabilities to roll:

(a) One six

**Solution:** Overall, there are $6^2 = 36$ possible outcomes.

Of these, 10 outcomes contain exactly one die roll of 6. Note we do not count a pair of sixes here.

Therefore, the probability of rolling a 6 on either die is $\frac{10}{36} = 0.2777$.

(b) A sum of 6

**Solution:** Of the $6^2 = 36$ possible outcomes, 5 outcomes that yield a sum of 6.

Therefore, the probability of rolling a sum of 6 on a pair of dice is $\frac{5}{36} = 0.13885$.

(c) A sum that is a multiple of 3

**Solution:** In this problem, we are looking for dice roll sums in set $S = \{3, 6, 9, 12\}$.

Of the $6^2 = 36$ possible outcomes, there are $2 + 5 + 4 + 1 = 12$ outcomes that yield a sum in set $S$. Therefore, the probability of rolling a sum that is a multiple of three on a pair of dice is $\frac{12}{36} = 0.3333$. 
9. [10 POINTS] Problem 15.27(a)-(b) [this problem is not in the F17 textbook]:
Draw two cards randomly from a 52-card deck. Compute the probabilities:

(a) The first is a $K$ and the second a picture-card (i.e., $A$, $K$, $Q$, $J$).

Solution: For the first card drawn, the probability of drawing a $K$ is

$$P[\text{first is } \text{K}] = \frac{4}{52} = 0.07692.$$ 

For the second card drawn, the probability of drawing a picture-card is

$$P[\text{second is picture-card|first is } \text{K}] = \frac{15}{51} = 0.2941.$$ 

Combining these two as a product, we have

$$P[\text{answer}] = \frac{4}{52} \times \frac{15}{51} = \frac{15}{663} = 0.02262.$$ 

(b) At least one card is a picture-card (i.e., $A$, $K$, $Q$, $J$).

Solution: Start with all possible ways to select two cards, i.e., $52 \times 51 = 2652$.

Next, determine all possible ways to select no picture-cards. Note that we have $52 - 16 = 36$ non-picture-cards.

Here, all possible ways to select two non-picture-cards is $36 \times 35 = 1260$.

Therefore, the probability of drawing two cards with at least one card being a picture-card is

$$1 - \frac{36 \times 35}{52 \times 51} = \frac{2652 - 1260}{2652} = 0.5249.$$ 

10. [10 POINTS] Problem 16.12(a)-(b) [this problem is not in the F17 textbook]: One in 20 men are colorblind and one in 400 women are colorblind. There are an equal number of men and women. You select a person at random.

(a) What is the probability that the person is colorblind?

Solution: Given that there are an equal number of men and women, we select each with probability $\frac{1}{2}$, which we then multiply and combine as follows:

$$\frac{1}{2} \times \frac{1}{20} + \frac{1}{2} \times \frac{1}{400} = \frac{1}{40} + \frac{1}{800} = \frac{21}{800} = 0.02625.$$ 

(b) The person is colorblind. What is the probability that the person is male?

Solution: Here, we are calculating $P[\text{person is male|person is colorblind}]$.

Using the conditional probability equation with $A$ representing “person is male” and $B$ representing “person is colorblind,” we have

$$P[A|B] = \frac{P[A \cap B]}{P[B]}.$$ 

First, $P[A \cap B] = \frac{20}{800} = 0.025$. And from (a), $P[B] = \frac{21}{800} = 0.02625$.

Then, we have

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{\frac{20}{800}}{\frac{21}{800}} = \frac{20}{21} = 0.9524.$$