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1 Problem 6.11

Prove by strong induction that \( n \leq 3^{\frac{n}{2}} \), for \( n \geq 0 \)

**Solution:**
We can rewrite the above predicate \( P(n) \) as \( 3^{\frac{n}{2}} \geq n \)
The above step can be taught of as a contraposition move?

On close inspection, this problem can only be solved by strong induction for a base case \( n \geq 4 \).
So our base case is \( n = 4 \), need to show that \( P(4) \) is True.
\[ P(4) = 3^{\frac{4}{2}} = 3 \times 3^{\frac{2}{2}} \approx 4.32 > 4 \]

Hence we have that \( P(4) \) holds.

We can now assume that \( P(4) \land P(5) \land P(6) \land \ldots \land P(n-1) \land P(n) \) is True.

Hence we can say that \( 3^{\frac{n}{2}} \geq n \)

We need to show that \( P(n + 1) \) is True, therefore need to prove that \( 3^{\frac{n+1}{2}} \geq (n+1) \)

We can rewrite \( 3^{\frac{n+1}{2}} \) as \( 3 \times 3^{\frac{n-1}{2}} \)

From strong induction we have that \( P(n - 2) \) is True, therefore we can say that \( 3^{\frac{n-2}{2}} \geq (n-2) \)

\( 3^{\frac{n+1}{2}} \geq 3 \times (n - 2) \)

\( 3^{\frac{n+1}{2}} \geq 3n - 6 \)

\( 3^{\frac{n+1}{2}} \geq n + 1 + (2n - 7) \)

But we know that for \( n \geq 4 \), \( 2n - 7 \geq 1 \)

Therefore we can conclude that \( 3^{\frac{n+1}{2}} \geq n + 1 \)

Hence proved

2 Problem 6.17

For \( x \in \mathbb{R} \), suppose \( x + \frac{1}{x} \in \mathbb{Z} \). Prove \( x^n + \frac{1}{x^n} \in \mathbb{Z} \) for \( n \geq 1 \).

Base case: \( P(1) \) is true because \( x + \frac{1}{x} \) is integer itself.

Assume \( P(1) \land \ldots \land P(n-1) \land P(n) \) is true. Then

\[ x^{k+1} + \frac{1}{x^{k+1}} = x^{k+1} + \frac{1}{x^{k+1}} + \frac{1}{x^k} - \frac{1}{x^k} \]

\[ = x^{k+1} + \frac{1}{x^{k-1}} + \frac{1}{x^{k+1}} - \frac{1}{x^k} \]

\[ = (x^k + \frac{1}{x^k}) \times x + \frac{1}{x^{k+1}} - \frac{1}{x^k} \]

\[ = (x^k + \frac{1}{x^k}) \times x + \frac{1}{x^k} - (x^k + \frac{1}{x^k}) \times \frac{1}{x} - \frac{1}{x^{k+1}} \]

\[ = (x^k + \frac{1}{x^k}) \times (x + \frac{1}{x}) - \frac{1}{x^{k+1}} + \frac{1}{x^{k+1}} - \frac{1}{x^k} \]

\[ = (x^k + \frac{1}{x^k}) \times (x + \frac{1}{x}) - (x^{k-1} + \frac{1}{x^{k-1}}) \]

From the assumption, \((x^k + \frac{1}{x^k}), (x + \frac{1}{x})\) and \((x^{k-1} + \frac{1}{x^{k-1}})\) are all integers,
so \( x^{k+1} + \frac{1}{x^{k+1}} \) is also integer, which proves \( P(n + 1) \)

So it follows by induction that \( P(n) \) is true for all \( n \geq 1 \), \( n \in \mathbb{Z} \)

3 Problem 7.4

3.1 \( A_0 = 0 \) and \( A_n = A_{n-1} + 1 \) for \( n \geq 1 \)

**Solution:**
To derive a formula, let’s write down the different values for \( A \) and see if we can deduce a pattern:
\( A_0 = 0 \)
\( A_1 = A_0 + 1 = 1 \)
\( A_2 = A_1 + 1 = 2 \)
\( A_3 = A_2 + 1 = 3 \)
\( A_4 = A_3 + 1 = 4 \)

By the above we can generalise that \( A_n = n \)
For the above to hold, we need to show this by induction:
Need to prove that $P(n) \rightarrow P(n+1)$ when $P(n) = n$

Base case, $n = 1$
For $n = 1$, $P(n) = 1 = n$. Therefore the base case of $n = 1$ holds.
We assume that $P(n)$ is True. We can say that $P(n) = n$ We need to show that $P(n + 1) = n + 1$
By the recursion we have that:
$A_{n+1} = A_n + 1$
But, $A_n = n = P(n)$
Hence, $A_{n+1} = n + 1$
Hence proved that $P(n) = n$; for $n \geq 1$

3.2 b $A_1 = 1$ and $A_2 = 2$ and $A_n = A_{n-1} + 2A_{n-2}$ for $n \geq 2$

Solution:
Like in the previous problem, let’s try and deduce a solution analytically by listing out the chain of values.
$A_3 = A_2 + 2A_1 = 2 + 2 \times 1 = 4$
$A_4 = A_3 + 2A_2 = 4 + 2 \times 2 = 8$
$A_5 = A_4 + 2A_3 = 8 + 2 \times 4 = 16$
$A_6 = A_5 + 2A_4 = 16 + 2 \times 8 = 32$
From the above it clearly looks like $P(n) = 2^{n-1}$; for $n \geq 2$
Let’s try and prove this claim of $P(n) = 2^{n-1}$ by Strong Induction.
Base case $n = 2$:
For $n = 2$, $P(2) = 2^{2-1} = 2$; hence the base case holds.
Since we are using strong induction we can assume that $P(2) \land P(3) \land P(n - 1) \land P(n)$ is True.
$P(n) = 2^{n-1}$ is hence true.
Need to prove that $P(n+1) = 2^n$
By the recursive definition we have that $P_{n+1} = P_n + 2P_{n-1}$
Since we are using strong induction we have that $P_n = 2^{n-1}$ and $P_{n-1} = 2^{n-2}$
Substituting back we get:
$P_{n+1} = 2^{n-1} + 2 \times 2^{n-2}$
$P_{n+1} = 2^{n-1} + 2^{n-2}$
$P_{n+1} = 2^{n-1}$
Thus proved that the claim $P(n) = 2^{n-1}$ holds.

4 Problem 7.8
Recall the Fibonacci numbers $F_1 = 1, F_2 = 1$; $F_n = F_{n-1} + F_{n-2}$ for $n > 2$

4.1 k Prove that $F_n$ and $F_{n+1}$ have no common factor except for 1; for $n \geq 1$

Solution:
We will use a proof of induction to prove the above:
Let’s assume a base case of $n = 1$, we need to show that $gcd(F_1, F_2) = 1$
We are given that $F_1 = 1$ and $F_2 = 1$; therefore $gcd(F_1, F_2) = gcd(1, 1) = 1$
Hence the claim holds for the base case of $n = 1$
We can assume that $P(n)$ is True, therefore $gcd(F_n, F_{n+1}) = 1$
We need to show that $P(n + 1)$ is True, therefore need to show that $gcd(F_{n+1}, F_{n+2}) = 1$
By the definition of the Fibonacci sequence, we have that $F_{n+1} = F_n + F_{n-1}$
And $F_{n+2} = F_{n+1} + F_n$
$gcd(F_{n+1}, F_{n+2}) = gcd(F_{n+1}, F_{n+1} + F_n)$
And by the Euclidean properties we have that $gcd(a, a + b) = gcd(a, a + b)$ because of the fact that any $d$ that divides both $a$ and $b$, $(a = kd, b = k'd)$ must also divide $a + b$ (because $a + b = (k + k')d$), and any $d$ that divides both $a$ and $a + b (a = xd, a + b = yd)$ must also divide $b$ (because $b = (y\tilde{x}d)$
Hence $gcd(F_{n+1}, F_n + F_{n+1}) = gcd(F_{n+1}, F_n)$
And this is given by $gcd(F_n, F_{n+1}) = 1$, which is True.
Hence proved.
5 Problem 7.31

Give recursive definitions for the set $S$ in the following case.

5.1 $S = \{1, 2, 3, 4, 6, 7, 8, 9, 11\}$, numbers which are not multiples of 5

Solution:
We can use leaping induction in this case, since we have multiple base cases and we can show that we can express numbers in the form of these as:

$P(1) \implies P(6) \implies P(11) \ldots \implies P((n - 4) + 5)$
$P(2) \implies P(7) \implies P(12) \ldots \implies P((n - 3) + 5)$
$P(3) \implies P(8) \implies P(13) \ldots \implies P((n - 2) + 5)$

$P(4) \implies P(9) \implies P(14) \ldots \implies P((n - 1) + 5)$

So clearly, we get the form $P(n) \implies P(n + k)$; for $k \in \{1, 2, 3, 4\}$

The recursive definition can be built from this as:

$\{1, 2, 3, 4\} \in S$ [basis]

$x \in S \implies x + 5 \in S$ [constructor]

6 Problem 8.2

Use structural induction to show that every nonempty string in the set $M$ of matched parentheses begins with an opening parenthesis [. Prove that $[] \in M$. Is every string that begins with an opening parenthesis in $M$?

Solution:
Let’s begin by writing down the definition for the recursive set of parentheses matched strings $M$.

$\epsilon \in M$ [basis]

$x, y \in M \implies [x]y \in M$

Let’s define $P(n)$ as $P(n) = \text{begin with } [$. The base case for a non-empty string in $M$ is $[]$.

Clearly this begins with an opening parentheses so we have proved our claim for the base case.

Assume that $P(1) \land P(2) \land P(3) \land P(4) \land P(5) \land P(n - 1) \land P(n)$ are True. This means that $P(n)$ begins with a $[$.

We need to prove that $P(n + 1)$ also begins with $[$.

We can compose $P(n + 1) = [s_k][s_l]$.

By strong induction we have that every element before $n + 1$ begins with $[$, hence $s_k$ and $s_l$ also begin with $[$.

Therefore we have that since we always concatenate a $[$ in the beginning, the $P(n + 1)$ also begins with $[.$

Having proved the claim $P(n)$; we are now in a position to tell whether $[]$ adheres to the claim. It clearly doesn’t as this element doesn’t begin with an opening parentheses, therefore it breaks the strong induction property we just proved.