Every American has a dream.

Kilam has some gray hair.
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Any map can be colored with 4 colors with adjacent countries having different colors.
Quantifiers

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Every even integer $n > 2$ is the sum of 2 primes (Goldbach, 1742).
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All cars have four wheels.
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These statements are more complex because of quantifiers:

\[ \text{EVERY; A; SOME; ANY; ALL; THERE EXISTS.} \]
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Compare:

My Ford Escort has four wheels;
ALL cars have four wheels.
ALL cars have four wheels
ALL cars have four wheels

Define predicate $P(c)$ and its domain
ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

$$C = \{c|c \text{ is a car}\} \quad \leftarrow \text{set of cars}$$
**ALL** cars have four wheels

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“for all \( c \) in \( C \), the statement \( P(c) \) is true.”

\[
\forall c \in C : P(c).
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(\( \forall \) means “for all”)

Creator: Malik Magdon-Ismail

Making Precise Statements: 21 / 25

There EXISTS →
ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

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Predicate Function

Creator: Malik Magdon-Ismail

Making Precise Statements: 21 / 25
Predicticates Are Like Functions

**ALL cars have four wheels**

Define *predicate* $P(c)$ and its *domain*

\[ C = \{c|c \text{ is a car}\} \quad \leftarrow \text{set of cars} \]

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Predicates Are Like Functions

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There **exists** a Creature with Blue eyes and Blonde Hair

Define *predicate* \( Q(a) \) and its *domain*

\[
A = \{ a | a \text{ is a creature} \} \quad \leftarrow \text{set of creatures}
\]
There exists a Creature with Blue eyes and Blonde Hair

Define predicate $Q(a)$ and its domain

$$A = \{a \mid a \text{ is a creature}\} \leftarrow \text{set of creatures}$$

$$Q(a) = "a \text{ has blue eyes and blonde hair}"$$
There EXISTS a Creature with Blue eyes and Blonde Hair

Define predicate $Q(a)$ and its domain

$$A = \{ a | a \text{ is a creature} \} \quad \leftarrow \text{set of creatures}$$

$$Q(a) = \text{“a has blue eyes and blonde hair”}$$

“there exists $a$ in $A$ for which the statement $Q(a)$ is true.”

$$\exists a \in A : Q(a).$$

($\exists$ means “there exists”)

Creator: Malik Magdon-Ismail
Making Precise Statements: 22 / 25
Negating Quantifiers →
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$G(a) = \text{“a has blue eyes”}$
There EXISTS a Creature with Blue eyes and Blonde Hair

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\exists a \in A : (G(a) \land H(a))
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(compound predicate)
There **exists** a Creature with Blue eyes and Blonde Hair

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(compound predicate)

(When the domain is understood, we don’t need to keep repeating it. We write $\exists a : Q(a)$, or $\exists a : (G(a) \land H(a))$.)
IT IS NOT THE CASE THAT (There is creature with blue eyes and blonde hair)
IT IS NOT THE CASE THAT (There is creature with blue eyes and blonde hair)

Same as: “All creatures don’t have blue eyes and blonde hair”

\[ -\left( \exists a \in A : Q(a) \right) \equiv \forall a \in A : \neg Q(a) \]
Negating Quantifiers

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**IT IS NOT THE CASE THAT** (All cars have four wheels)
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\[-(\exists a \in A : Q(a)) \equiv \forall a \in A : \neg Q(a)\]

IT IS NOT THE CASE THAT (All cars have four wheels)
Same as: “There is a car which does not have four wheels”

\[-(\forall c \in C : P(c)) \equiv \exists c \in C : \neg P(c)\]
Negating Quantifiers

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When you take the negation inside the quantifier and negate the predicate, you must switch quantifiers: \( \forall \rightarrow \exists \), \( \exists \rightarrow \forall \)
Every American Has a Dream

Define domains and a predicate.

\[ A = \{ a \mid a \text{ is an American}\}; \]
\[ D = \{ d \mid d \text{ is a dream}\}. \]
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\[ A = \{a \mid a \text{ is an American}\}; \]
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\[ P(a, d) = \text{“American } a \text{ has dream } d.” \]
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- There is some special dream \( d \), and every American \( a \) has that dream.
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\[ P(a, d) = \text{“American } a \text{ has dream } d.” \]

There is some special dream \( d \), and every American \( a \) has that dream.

\[ \exists d \in D : (\forall a \in A : P(a, d)). \]
Every American Has a Dream

Define domains and a predicate.

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- For every American \( a \), they have there own private dream \( d \).
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- For every American \( a \), they have their own private dream \( d \).
  \[ \forall a \in A : (\exists d \in D : P(a, d)). \]

When quantifiers are mixed, the order in which they appear is important for the meaning. Order generally cannot be switched.
Proofs with Quantifiers

Claim 1. \( \forall n > 2 : \text{IF } n \text{ is even, THEN } n \text{ is a sum of two primes.} \) (Goldbach, 1742)

Claim 2. \( \exists (a, b, c) \in \mathbb{N}^3 : a^2 + b^2 = c^2. \) \((a, b, c) \in \mathbb{N}^3 \text{ means triples of natural numbers})

Claim 3. \( \neg \exists (a, b, c) \in \mathbb{N}^3 : a^3 + b^3 = c^3. \)

Claim 4. \( \forall (a, b, c) \in \mathbb{N}^3 : a^3 + b^3 \neq c^3. \)

Think about what it would take to prove these claims.