Today: Induction, Proving “...for all...”

1. What is induction.

2. Why do we need it?

3. The principle of induction. Toppling the dominoes. The induction template.

4. Examples.

5. Induction and Well-Ordering.
Dispensing Postage Using 5¢ and 7¢ Stamps

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Creator: Malik Magdon-Ismail

Induction: Proving “For All . . .”: 4 / 18

Why Induction? →
## Dispensing Postage Using 5¢ and 7¢ Stamps

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Can every postage greater than 23¢ can be dispensed?
Dispensing Postage Using 5¢ and 7¢ Stamps

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Can every postage greater than 23¢ can be dispensed?

Intuitively yes.
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Can every postage greater than 23¢ can be dispensed?

Intuitively yes.

**Induction** is the formalization of that intuition.
# Why Do We Need Induction?

## Predicate Claim

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**Induction.** Systematic.
Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

$P(n) = "4^n - 1$ is divisible by 3."
Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

$P(n) = "4^n - 1$ is divisible by 3."

We proved:

\[ \text{IF } 4^n - 1 \text{ is divisible by 3, THEN } 4^{n+1} - 1 \text{ is divisible by 3.} \]

\[ P(n) \quad P(n+1) \]

\textit{Proof.} We prove the claim using a direct proof.

1: Assume that $P(n)$ is T, that is $4^n - 1$ is divisible by 3.
2: This means that $4^n - 1 = 3k$ for an integer $k$, or that $4^n = 3k + 1$.
3: Observe that $4^{n+1} = 4 \cdot 4^n$, and since $4^n = 3k + 1$, it follows that
\[ 4^{n+1} = 4 \cdot (3k + 1) = 12k + 4. \]

Therefore $4^{n+1} - 1 = 12k + 3 = 3(4k + 1)$ is a multiple of 3 ($4k + 1$ is an integer).
4: Since $4^{n+1} - 1$ is a multiple of 3, we have shown that $4^{n+1} - 1$ is divisible by 3.
5: Therefore, $P(n + 1)$ is T.
Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

$P(n) = "4^n - 1 \text{ is divisible by 3}."

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$P(n) \rightarrow P(n + 1)$
Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

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What use is this? We can’t answer the question. (Reasoning in the absense of facts.)
$4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = \text{“}4^n - 1 \text{ is divisible by 3.”}$

$P(n) \rightarrow P(n + 1)$
$4^n - 1$ is Divisible by 3 for $n \geq 1$

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P(n) = "4^n - 1 \text{ is divisible by 3.}"
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P(n) \rightarrow P(n + 1)
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From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3$ \hspace{2cm} \leftarrow \text{divisible by 3 (new fact)}

$P(1)$
$4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = "4^n - 1$ is divisible by 3."

$P(n) \rightarrow P(n+1)$

From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3$  $\leftarrow$ divisible by 3 (new fact)

$P(1) \rightarrow P(2)$
4^n - 1 is Divisible by 3 for n ≥ 1

\[ P(n) = \text{“}4^n - 1 \text{ is divisible by 3.”} \]

\[ P(n) \rightarrow P(n + 1) \]

From tinkering we know that \( P(1) \) is T: \( 4^1 - 3 = 3 \) divisible by 3 (new fact)

\[ P(1) \rightarrow P(2) \]
4^n - 1 is Divisible by 3 for \( n \geq 1 \)

\[ P(n) = \text{"}4^n - 1 \text{ is divisible by 3."} \]

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$\checkmark \quad \checkmark \quad \checkmark \quad \checkmark$

$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4)$
$4^n - 1$ is Divisible by 3 for $n \geq 1$

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$P(n) \rightarrow P(n + 1)$

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$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4)$
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$P(n) = "4^n - 1$ is divisible by 3."

$P(n) \rightarrow P(n + 1)$

From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3 \leftarrow$ divisible by 3 (new fact)

$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n - 1)$
$4^n - 1$ is Divisible by 3 for $n \geq 1$

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$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n-1) \rightarrow P(n)$
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\[ P(n) \rightarrow P(n + 1) \]

From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3$ \hspace{1cm} $\leftarrow$ divisible by 3 (new fact)

$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n - 1) \rightarrow P(n) \rightarrow \cdots$
By Induction, $4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = "4^n - 1 \text{ is divisible by 3.}"

1. $P(1)$ is T.✓
2. $P(n) \rightarrow P(n+1)$ is T.✓

By induction, $P(n)$ is T for all $n \geq 1$. 
By Induction, $4^n - 1$ is Divisible by 3 for $n \geq 1$

$P(n) = "4^n - 1$ is divisible by 3."

1. $P(1)$ is T. ✓
2. $P(n) \rightarrow P(n + 1)$ is T. ✓

By induction, $P(n)$ is T for all $n \geq 1$.

$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots$

Practice. Exercise 5.2.

$P(n)$ form an infinite chain of dominos. Topple the first and they all fall.
Induction to prove: $\forall n \geq 1 : P(n)$.

Proof. We use induction to prove $\forall n \geq 1 : P(n)$. 
Induction to prove: \( \forall n \geq 1 : P(n) \).

Proof. We use induction to prove \( \forall n \geq 1 : P(n) \).

1: Show that \( P(1) \) is T. ("simple" verification.) [base case]
**Induction to prove:** $\forall n \geq 1 : P(n)$.

**Proof.** We use induction to prove $\forall n \geq 1 : P(n)$.

1. Show that $P(1)$ is T. (“simple” verification.)  
   [base case]
2. Show $P(n) \rightarrow P(n+1)$ for $n \geq 1$  
   [induction step]
**Induction to prove:** $\forall n \geq 1 : P(n)$.

**Proof.** We use induction to prove $\forall n \geq 1 : P(n)$.

1. Show that $P(1)$ is T. ("simple" verification.)
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*[base case]*

*[induction step]*
Induction to prove: $\forall n \geq 1 : P(n)$.

Proof. We use induction to prove $\forall n \geq 1 : P(n)$.

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2: Show $P(n) \rightarrow P(n+1)$ for $n \geq 1$

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3: Conclude: by induction, $\forall n \geq 1 : P(n)$.  

\qed
Induction to prove: \( \forall n \geq 1 : P(n) \).

**Proof.** We use induction to prove \( \forall n \geq 1 : P(n) \).

1. Show that \( P(1) \) is T. ("simple" verification.)  
2. Show \( P(n) \rightarrow P(n+1) \) for \( n \geq 1 \)  

   **Prove the implication** using direct proof or contraposition.

   **Direct**
   - Assume \( P(n) \) is T.  
   - (valid derivations)
   - must show for any \( n \geq 1 \)
   - must use \( P(n) \) here
   - **Show** \( P(n+1) \) is T.

   **Contraposition**
   - Assume \( P(n+1) \) is F.  
   - (valid derivations)
   - must show for any \( n \geq 1 \)
   - must use \( \neg P(n+1) \) here

3. Conclude: by induction, \( \forall n \geq 1 : P(n) \).

- Prove the *implication* \( P(n) \rightarrow P(n+1) \) for a *general* \( n \geq 1 \). (Often direct proof)
  Why is this easier than just proving \( P(n) \) for general \( n \)?
- Assume \( P(n) \) is T, and reformulate it mathematically.
- Somewhere in the proof you *must* use \( P(n) \) to prove \( P(n+1) \).
- End with a statement that \( P(n+1) \) is T.
1 + 2 + 3 + \cdots + (n - 1) + n = ?
The GREAT Gauss (age 8-10):

\[
S(n) = 1 + 2 + 3 + \cdots + (n - 1) + n \\
\]

\[
S(n) = n + n - 1 + \cdots + 1 \\
\]

\[
2S(n) = (n + 1) + (n + 1) + \cdots + (n + 1) \\
= n \times (n + 1)
\]
The GREAT Gauss (age 8-10):

\[ S(n) = 1 + 2 + \cdots + n \]
\[ S(n) = n + n - 1 + \cdots + 1 \]
\[ 2S(n) = (n + 1) + (n + 1) + \cdots + (n + 1) \]
\[ = n \times (n + 1) \]

\[ S(n) = 1 + 2 + 3 + \cdots + (n - 1) + n = \frac{1}{2}n(n + 1) \]
Proof: \[ \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \]

Proof. (By Induction) \( P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \)

Proof. (By Induction) \( P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).

1: [Base case] \( P(1) \) claims that \( 1 = \frac{1}{2} \times 1 \times (1 + 1) \), which is clearly true.
Proof. (By Induction) \( P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).

1: **[Base case]** \( P(1) \) claims that \( 1 = \frac{1}{2} \times 1 \times (1 + 1) \), which is clearly \( \top \).

2: **[Induction step]** We show \( P(n) \to P(n + 1) \) for all \( n \geq 1 \), using a direct proof.

   Assume (induction hypothesis) \( P(n) \) is \( \top \): \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).

   Show \( P(n + 1) \) is \( \top \): \( \sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 1 + 1) \).
Proof: $\sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$

**Proof.** (By Induction) $P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$.

1: **[Base case]** $P(1)$ claims that $1 = \frac{1}{2} \times 1 \times (1 + 1)$, which is clearly T.

2: **[Induction step]** We show $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$, using a direct proof.

Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} i = \frac{1}{2}n(n + 1)$.

Show $P(n + 1)$ is T: $\sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 1 + 1)$.

\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1)
\]

Key step
Proof: \[ \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \]

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Show \( P(n + 1) \) is \( T \): \( \sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 1 + 1) \).

\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) \quad \text{Key step}
\]
\[
= \frac{1}{2}n(n + 1) + (n + 1) \quad \text{[by the induction hypothesis} \ P(n) \text{]}]
\]
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \)

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   Assume (induction hypothesis) \( P(n) \) is T: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).
   Show \( P(n + 1) \) is T: \( \sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 1 + 1) \).

\[
\begin{align*}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n + 1) \\
&= \frac{1}{2}n(n + 1) + (n + 1) \quad \text{[by the induction hypothesis \( P(n) \)]} \\
&= (n + 1)\left(\frac{1}{2}n + 1\right) \\
&= \frac{1}{2}(n + 1)(n + 2) = \frac{1}{2}(n + 1)(n + 1 + 1).
\end{align*}
\]
Proof. (By Induction) \( P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).

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2: [Induction step] We show \( P(n) \rightarrow P(n + 1) \) for all \( n \geq 1 \), using a direct proof. Assume (induction hypothesis) \( P(n) \) is true: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \).

Show \( P(n + 1) \) is true: \( \sum_{i=1}^{n+1} i = \frac{1}{2}(n + 1)(n + 1 + 1) \).

\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) \\
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= (n + 1) \left( \frac{1}{2}n + 1 \right) \\
= \frac{1}{2}(n + 1)(n + 2) = \frac{1}{2}(n + 1)(n + 1 + 1).
\]

This is exactly what was to be shown. So, \( P(n + 1) \) is true.
Proof: \( \sum_{i=1}^{n} i = \frac{1}{2}n(n + 1) \)

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\[
\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n + 1) = \frac{1}{2}n(n + 1) + (n + 1) = (n + 1)\left(\frac{1}{2}n + 1\right) = \frac{1}{2}(n + 1)(n + 2) = \frac{1}{2}(n + 1)(n + 1 + 1).
\]

This is exactly what was to be shown. So, \( P(n + 1) \) is true.

3: By induction, \( P(n) \) is true for all \( n \geq 1 \).

\[ \blacksquare \]
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 =? \]

Where’s the GREAT Gauss when you need him?
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 =? \]

Replace Gauss with TINKERING: \textit{method of differences}.

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<td>9</td>
<td>11</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>3rd difference</td>
<td>( S'''(n) )</td>
<td>2</td>
<td>2</td>
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\[ \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \rightarrow \]
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 =? \]

Replace Gauss with TINKERING: method of differences.

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3’rd difference constant is like 3’rd derivative constant.
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 =? \]

Replace Gauss with TINKERING: \textit{method of differences}.

\[ \begin{array}{c|cccccccc}
   n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
   \hline
   S(n) & 1 & 5 & 14 & 30 & 55 & 91 & 140 \\
   1st difference & S'(n) & 4 & 9 & 16 & 25 & 36 & 49 \\
   2nd difference & S''(n) & & 5 & 7 & 9 & 11 & 13 \\
   3rd difference & S'''(n) & & & 2 & 2 & 2 & 2 \\
\end{array} \]

3’rd difference constant is like 3’rd derivative constant. So guess:

\[ S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3. \]
Sum of Integer Squares

\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 =? \]

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3’rd difference constant is like 3’rd derivative constant. So guess:

\[ S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3. \]

\[
\begin{align*}
a_0 + a_1 + a_2 + a_3 &= 1 \\
a_0 + 2a_1 + 4a_2 + 8a_3 &= 5 \\
a_0 + 3a_1 + 9a_2 + 27a_3 &= 14 \\
a_0 + 4a_1 + 16a_2 + 64a_3 &= 30
\end{align*}
\]

\[
a_0 = 0, \ a_1 = \frac{1}{6}, \ a_2 = \frac{1}{2}, \ a_3 = \frac{1}{3}
\]

\[ \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1) \to \]
\[ S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 + n^2 =? \]

Replace Gauss with TINKERING: \textit{method of differences}.

\begin{align*}
\begin{array}{c|cccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
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S(n) & 1 & 5 & 14 & 30 & 55 & 91 & 140 \\
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\end{align*}

3′rd difference constant is like 3′rd derivative constant. So guess:

\[
S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3.
\]

\[
\begin{array}{c|cccccccccc}
 n \frac{1}{6} n + \frac{1}{2} n^2 + \frac{1}{3} n^3 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline
a_0 & 0 & \frac{1}{6} & 1 & \frac{1}{2} & \frac{1}{3} \\
a_1 & \frac{1}{6} & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\
a_2 & \frac{1}{12} & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \\
a_3 & \frac{1}{24} & \frac{1}{12} & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\
\end{array}
\]

\[
\begin{align*}
\sum_{i=1}^{n} i^2 &= \frac{1}{6} n(n + 1)(2n + 1) \\
a_0 + a_1 + a_2 + a_3 &= 1 \\
a_0 + 2a_1 + 4a_2 + 8a_3 &= 5 \\
a_0 + 3a_1 + 9a_2 + 27a_3 &= 14 \\
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a_0 &= 0, \ a_1 = \frac{1}{6}, \ a_2 = \frac{1}{2}, \ a_3 = \frac{1}{3}
\end{align*}
\]
Proof: \[ S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n + 1)(2n + 1) \]

Proof. (By induction.) \[ P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n + 1)(2n + 1). \]
Proof: \[ S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n + 1)(2n + 1) \]

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   Show \( P(n + 1) \) is \( T \): \( \sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n + 1)(n + 2)(2n + 3) \).

   \[
   \sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n + 1)^2 \quad \text{Key step}
   \]
Proof: \( S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6} n + \frac{1}{2} n^2 + \frac{1}{3} n^3 = \frac{1}{6} n(n + 1)(2n + 1) \)

Proof. (By induction.) \( P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n + 1)(2n + 1) \).

1: \textbf{[Base case]} \( P(1) \), claims that \( 1 = \frac{1}{6} \times 1 \times 2 \times 3 \), which is clearly \( \top \).

2: \textbf{[Induction step]} Show \( P(n) \implies P(n + 1) \) for all \( n \geq 1 \). Direct proof. Assume \( \text{(induction hypothesis)} P(n) \) is \( \top \): \( \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n + 1)(2n + 1) \).

Show \( P(n + 1) \) is \( \top \): \( \sum_{i=1}^{n+1} i^2 = \frac{1}{6} (n + 1)(n + 2)(2n + 3) \).

\[
\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n + 1)^2 \quad \text{Key step}
\]

\[
= \frac{1}{6} n(n + 1)(2n + 1) + (n + 1)^2 \quad \text{[by the induction hypothesis } P(n) \text{]}
\]
Proof. (By induction.) \( P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n + 1)(2n + 1) \).

1. **[Base case]** \( P(1) \), claims that \( 1 = \frac{1}{6} \times 1 \times 2 \times 3 \), which is clearly true.

2. **[Induction step]** Show \( P(n) \rightarrow P(n + 1) \) for all \( n \geq 1 \). Direct proof.

Assume (induction hypothesis) \( P(n) \) is true: \( \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n + 1)(2n + 1) \).

Show \( P(n + 1) \) is true: \( \sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n + 1)(n + 2)(2n + 3) \).

\[
\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n + 1)^2 \\
= \frac{1}{6}n(n + 1)(2n + 1) + (n + 1)^2 \\
= \frac{1}{6}(n + 1)(n + 2)(2n + 3)
\]

This is exactly what was to be shown. So, \( P(n + 1) \) is true.
Proof. (By induction.) \( P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n + 1)(2n + 1) \).

1: [**Base case**] \( P(1) \), claims that \( 1 = \frac{1}{6} \times 1 \times 2 \times 3 \), which is clearly \( T \).

2: [**Induction step**] Show \( P(n) \rightarrow P(n + 1) \) for all \( n \geq 1 \). Direct proof. Assume (induction hypothesis) \( P(n) \) is \( T \): \( \sum_{i=1}^{n} i^2 = \frac{1}{6} n(n + 1)(2n + 1) \).

Show \( P(n + 1) \) is \( T \): \( \sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n + 1)(n + 2)(2n + 3) \).

\[
\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n + 1)^2 \\
= \frac{1}{6} n(n + 1)(2n + 1) + (n + 1)^2 \\
[\text{by the induction hypothesis } P(n)] \\
= \frac{1}{6}(n + 1)(n + 2)(2n + 3)
\]

This is exactly what was to be shown. So, \( P(n + 1) \) is \( T \).

3: By induction, \( P(n) \) is \( T \ \forall n \geq 1. \)

\[\square\]
Well-ordering Principle.

Any non-empty set of natural numbers has a minimum element.
Well Ordering Principle

Well-ordering Principle.

Any non-empty set of natural numbers has a minimum element.

Induction follows from well ordering.
Well-ordering Principle.

Any non-empty set of natural numbers has a minimum element.

Induction follows from well ordering. Let $P(1)$ and $P(n) \rightarrow P(n + 1)$ be true.
Well Ordering Principle

Well-ordering Principle.

*Any* non-empty set of natural numbers has a minimum element.

Induction follows from well ordering. Let $P(1)$ and $P(n) \rightarrow P(n + 1)$ be T.

Suppose $P(n_*)$ fails for the **smallest** counter-example $n_*$ (well-ordering).
Well Ordering Principle

**Well-ordering Principle.**

*Any* non-empty set of natural numbers has a minimum element.

Induction follows from well ordering. Let $P(1)$ and $P(n) \rightarrow P(n + 1)$ be T.

Suppose $P(n_*)$ fails for the **smallest** counter-example $n_*$ (well-ordering).

\[
\begin{align*}
P(1) & \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n_* - 1) \rightarrow P(n_*) \rightarrow \cdots
\end{align*}
\]
Well Ordering Principle

Well-ordering Principle.
Any non-empty set of natural numbers has a minimum element.

Induction follows from well ordering. Let $P(1)$ and $P(n) \rightarrow P(n + 1)$ be T.

Suppose $P(n_*)$ fails for the smallest counter-example $n_*$ (well-ordering).

$$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n_* - 1) \rightarrow P(n_*) \rightarrow \cdots$$

Now how can $P(n_* - 1) \rightarrow P(n_*)$ be T?
Well Ordering Principle

**Well-ordering Principle.**

*Any* non-empty set of natural numbers has a minimum element.

Induction follows from well ordering. Let $P(1)$ and $P(n) \rightarrow P(n + 1)$ be true.

Suppose $P(n_*)$ fails for the *smallest* counter-example $n_*$ (well-ordering).

\[
P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots \rightarrow P(n_* - 1) \rightarrow P(n_*) \rightarrow \cdots
\]

Now how can $P(n_* - 1) \rightarrow P(n_*)$ be true?

Any induction proof can also be done using well-ordering.
Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

Proof. [Induction] \( P(n) : n < 2^n \).
Proof. [Induction] \( P(n) : n < 2^n \).

Base case. \( P(1) \) is true because \( 1 < 2^1 \).
Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

Proof. [Induction] $P(n) : n < 2^n$.

Base case. $P(1)$ is $t$ because $1 < 2^1$.

Induction. Assume $P(n)$ is $t$: $n < 2^n$. and show $P(n + 1)$ is $t$: $n + 1 < 2^{n+1}$.

\[ n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}. \]
Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

Proof. [Induction] $P(n) : n < 2^n$.

**Base case.** $P(1)$ is T because $1 < 2^1$.

**Induction.** Assume $P(n)$ is T: $n < 2^n$. and show $P(n + 1)$ is T: $n + 1 < 2^{n+1}$.

\[
n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
\]

Therefore $P(n + 1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$. 

\[\blacksquare\]
Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

Proof. [Induction] $P(n) : n < 2^n$.

Base case. $P(1)$ is T because $1 < 2^1$.

Induction. Assume $P(n)$ is T: $n < 2^n$. and show $P(n + 1)$ is T: $n + 1 < 2^{n+1}$.

$$n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.$$ 

Therefore $P(n + 1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$.

Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

Proof. [Induction] $P(n) : n < 2^n$.

Base case. $P(1)$ is T because $1 < 2^1$.

Induction. Assume $P(n)$ is T: $n < 2^n$. and show $P(n + 1)$ is T: $n + 1 < 2^{n+1}$.

$$n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.$$ 

Therefore $P(n + 1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$.


Assume that there is an $n \geq 1$ for which $n \geq 2^n$. 

Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

**Proof.** [Induction] \( P(n) : n < 2^n \).

**Base case.** \( P(1) \) is T because \( 1 < 2^1 \).

**Induction.** Assume \( P(n) \) is T: \( n < 2^n \) and show \( P(n + 1) \) is T: \( n + 1 < 2^{n+1} \).

\[
n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
\]

Therefore \( P(n + 1) \) is T and, by induction, \( P(n) \) is T for \( n \geq 1 \).

**Proof.** [Well-ordering] Proof by contradiction.

Assume that there is an \( n \geq 1 \) for which \( n \geq 2^n \).

Let \( n^* \) be the minimum such counter-example, \( n^* \geq 2^{n^*} \).
Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

**Proof.** [Induction] \( P(n) : n < 2^n \).

**Base case.** \( P(1) \) is T because \( 1 < 2^1 \).

**Induction.** Assume \( P(n) \) is T: \( n < 2^n \). and show \( P(n+1) \) is T: \( n + 1 < 2^{n+1} \).

\[
n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
\]

Therefore \( P(n+1) \) is T and, by induction, \( P(n) \) is T for \( n \geq 1 \).

**Proof.** [Well-ordering] Proof by **contradiction**.

Assume that there is an \( n \geq 1 \) for which \( n \geq 2^n \).

Let \( n_* \) be the **minimum** such **counter-example**, \( n_* \geq 2^{n_*} \).

Since \( 1 < 2^1 \), \( n_* \geq 2 \). Since \( n_* \geq 2 \), \( \frac{1}{2} n_* \geq 1 \) and so,

\[
n_* - 1 \geq n_* - \frac{1}{2} n_* = \frac{1}{2} n_* \geq \frac{1}{2} \times 2^{n_*} = 2^{n_*-1}.
\]
Example Well-Ordering Proof: \( n < 2^n \) for \( n \geq 1 \)

**Proof.** [Induction] \( P(n) : n < 2^n \).

**Base case.** \( P(1) \) is T because \( 1 < 2^1 \).

**Induction.** Assume \( P(n) \) is T: \( n < 2^n \) and show \( P(n + 1) \) is T: \( n + 1 < 2^{n+1} \).

\[
    n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
\]

Therefore \( P(n + 1) \) is T and, by induction, \( P(n) \) is T for \( n \geq 1 \).

**Proof.** [Well-ordering] Proof by **contradiction**.
Assume that there is an \( n \geq 1 \) for which \( n \geq 2^n \).
Let \( n^*_\) be the **minimum** such **counter-example**, \( n^*_i = 2^{n^*_i} \).
Since \( 1 < 2^1 \), \( n^*_i \geq 2 \). Since \( n^*_i \geq 2 \), \( \frac{1}{2} n^*_i \geq 1 \) and so,

\[
    n^*_i - 1 \geq n^*_i - \frac{1}{2} n^*_i = \frac{1}{2} n^*_i \geq \frac{1}{2} \times 2^{n^*_i} = 2^{n^*_i-1}.
\]

So, \( n^*_i - 1 \) is a **smaller** counter example. **FISHY!**
Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

**Proof.** [Induction] $P(n) : n < 2^n$.

**Base case.** $P(1)$ is T because $1 < 2^1$.

**Induction.** Assume $P(n)$ is T: $n < 2^n$. and show $P(n + 1)$ is T: $n + 1 < 2^{n+1}$.

\[
n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.
\]

Therefore $P(n + 1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$.

---

**Proof.** [Well-ordering] Proof by **contradiction**.

Assume that there is an $n \geq 1$ for which $n \geq 2^n$.

Let $n_*$ be the **minimum** such **counter-example**, $n_* \geq 2^{n_*}$.

Since $1 < 2^1$, $n_* \geq 2$. Since $n_* \geq 2$, $\frac{1}{2}n_* \geq 1$ and so,

\[
n_* - 1 \geq n_* - \frac{1}{2}n_* = \frac{1}{2}n_* \geq \frac{1}{2} \times 2^{n_*} = 2^{n_*-1}.
\]

So, $n_* - 1$ is a **smaller** counter example. **FISHY!**

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The **method of minimum counter-example** is very powerful.
Practice. Problems 5.*