Today: Conditional Probability

1. New information changes a probability.

2. Definition of conditional probability from regular probability.

3. Conditional probability traps
   - Sampling bias.
   - Transposed conditional.

4. Law of total probability.
   - Probabilistic case-by-case analysis.
Flu Season

1. Chances a random person has the flu is about 0.01 (or 1%) \((prior\) probability).

   Probability of flu: \(P[\text{flu}] \approx 0.01\).

2. You have a slight fever – \textit{new information}. Chances of flu “increase”.

   Probability of flu \textit{given} fever: \(P[\text{flu} | \text{fever}] \approx 0.4\).

   - New information changes the prior probability to the \textit{posterior} probability.
   - Translate posterior as “\textit{After} you get the new information.”

   \(P[A | B]\) is the (updated) \textit{conditional} probability of \(A\), \textit{given} the new information \(B\).

3. Roommie has flu (more new information). Flu for sure, take counter-measures.

   Probability of flu \textit{given} fever and roommie flu: \(P[\text{flu} | \text{fever AND roommie flu}] \approx 1\).
5,000 students: 1,000 CS; 100 MATH; 80 dual MATH-CS.

Pick a random student:

\[ P[\text{CS}] = \frac{1000}{5000} = 0.2; \]
\[ P[\text{MATH}] = \frac{100}{5000} = 0.02; \]
\[ P[\text{CS AND MATH}] = \frac{80}{5000} = 0.016. \]
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\]

New information: student is MATH. What is \( P[\text{CS} | \text{MATH}] \)?

- Effectively picking a random student from MATH.
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- Effectively picking a random student from MATH.
- New probability of CS \( \sim \) striped area \( |CS \cap MATH| \).
5,000 students: 1,000 CS; 100 MATH; 80 dual MATH-CS.

Pick a random student:

\[
\begin{align*}
P[CS] &= \frac{1000}{5000} = 0.2; \\
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\end{align*}
\]

New information: student is MATH. What is \(P[CS | MATH]\)?
- Effectively picking a random student from MATH.
- New probability of CS \sim stripped area \(|CS \cap MATH|\).

\[
P[CS | MATH] = \frac{|CS \cap MATH|}{|MATH|} = \frac{80}{100} = 0.8.
\]

MATH students are 4 times more likely to be CS majors than a random student.

**Pop Quiz.** What is \(P[MATH | CS]\)? What is \(P[CS | CS \text{ OR MATH}]\)?
Conditional Probability $\mathbb{P}[A \mid B]$

$\mathbb{P}[A \mid B] = \text{frequency of outcomes known to be in } B \text{ that are also in } A$.
Conditional Probability \( \mathbb{P}[A \mid B] \)

\[ \mathbb{P}[A \mid B] = \text{frequency of outcomes known to be in } B \text{ that are also in } A. \]

\( n_B \) outcomes in event \( B \) when you repeat an experiment \( n \) times.

\[
\mathbb{P}[B] = \frac{n_B}{n}.
\]
Conditional Probability $\mathbb{P}[A \mid B]$

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$n_B$ outcomes in event $B$ when you repeat an experiment $n$ times.

$$\mathbb{P}[B] = \frac{n_B}{n}.$$  

Of the $n_B$ outcomes in $B$, the number also in $A$ is $n_{A \cap B}$,

$$\mathbb{P}[A \cap B] = \frac{n_{A \cap B}}{n}.$$  

Conditional Probability $\mathbb{P}[A \mid B]$

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$n_B$ outcomes in event $B$ when you repeat an experiment $n$ times.

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Of the $n_B$ outcomes in $B$, the number also in $A$ is $n_{A \cap B}$,

$$\mathbb{P}[A \cap B] = \frac{n_{A \cap B}}{n}.$$  

The frequency of outcomes in $A$ among those outcomes in $B$ is $n_{A \cap B}/n_B$,

$$\mathbb{P}[A \mid B] = \frac{n_{A \cap B}}{n_B} = \frac{n_{A \cap B}}{n} \times \frac{n}{n_B} = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}.$$
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It is cloudy one in five days, $\Pr[\text{Clouds}] = \frac{1}{5}$. It rains one in seven days, $\Pr[\text{Rain}] = \frac{1}{7}$. 

It is cloudy one in five days, \( \mathbb{P}[\text{Clouds}] = \frac{1}{5} \). It rains one in seven days, \( \mathbb{P}[\text{Rain}] = \frac{1}{7} \).

What are the chances of rain on a cloudy day?

\[
\mathbb{P}[\text{Rain} \mid \text{Clouds}] = \frac{\mathbb{P}[\text{Rain} \cap \text{Clouds}]}{\mathbb{P}[\text{Clouds}]}.
\]
Chances of Rain Given Clouds

It is cloudy one in five days, $\Pr[\text{Clouds}] = \frac{1}{5}$. It rains one in seven days, $\Pr[\text{Rain}] = \frac{1}{7}$.

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$\{\text{Rainy Days}\} \subseteq \{\text{Cloudy Days}\} \rightarrow \Pr[\text{Rain} \cap \text{Clouds}] = \Pr[\text{Rain}]$. 

{Rainy Days} ⊆ {Cloudy Days} → $\Pr[\text{Rain} \cap \text{Clouds}] = \Pr[\text{Rain}]$. 

Creator: Malik Magdon-Ismail
Conditional Probability: 7 / 16 
Conditioning with Dice →
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\]

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\[
P[\text{Rain} \mid \text{Clouds}] = \frac{P[\text{Rain}]}{P[\text{Clouds}]} = \frac{\frac{1}{7}}{\frac{1}{5}} = \frac{5}{7}.
\]

5-times more likely to rain on a cloudy day than on a random day.

Crucial first step: identify the conditional probability. What is the “new information”? 
Two dice have both rolled odd. What are the chances the sum is 10?

\[
P[\text{Sum is 10} \mid \text{Both are Odd}] = \frac{P[(\text{Sum is 10}) \text{ AND } (\text{Both are Odd})]}{P[\text{Both are Odd}]}\]
Two dice have both rolled odd. What are the chances the sum is 10?

\[
P[\text{Sum is 10} \mid \text{Both are Odd}] = \frac{P[(\text{Sum is 10}) \ \text{AND} \ (\text{Both are Odd})]}{P[\text{Both are Odd}]}
\]

### Probability Space

Die 1 Value

- 1
- 2
- 3
- 4
- 5
- 6

Die 2 Value

- 1
- 2
- 3
- 4
- 5
- 6

Creator: Malik Magdon-Ismail

Conditional Probability: 8 / 16

Computing a Conditional Probability →
Two dice have both rolled odd. What are the chances the sum is 10?

\[
P[\text{Sum is 10} \mid \text{Both are Odd}] = \frac{P[(\text{Sum is 10}) \text{ AND (Both are Odd)}]}{P[\text{Both are Odd}]}\]

Probability Space

<table>
<thead>
<tr>
<th>Die 1 Value</th>
<th>Die 2 Value</th>
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<tbody>
<tr>
<td>□□</td>
<td>1/36</td>
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Creator: Malik Magdon-Ismail  
Conditional Probability: 8/16
Two dice have both rolled odd. What are the chances the sum is 10?

\[
P[\text{Sum is 10} \mid \text{Both are Odd}] = \frac{P[(\text{Sum is 10}) \text{ AND } (\text{Both are Odd})]}{P[\text{Both are Odd}]}\]

\[
P[\text{Sum is 10}] = \frac{3}{36} = \frac{1}{12}.
\]

**Probability Space**

<table>
<thead>
<tr>
<th>Die 1 Value</th>
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<tbody>
<tr>
<td>1(\frac{1}{36})</td>
<td>2(\frac{1}{36})</td>
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<td>1(\frac{1}{36})</td>
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</table>

Creator: Malik Magdon-Ismail
\( \mathbb{P}[\text{Sum of 2 Dice is 10 } | \text{ Both are Odd}] \)

Two dice have both rolled odd. What are the chances the sum is 10?

\[
\mathbb{P}[\text{Sum is 10 } | \text{ Both are Odd}] = \frac{\mathbb{P}[(\text{Sum is 10}) \text{ AND (Both are Odd)}]}{\mathbb{P}[\text{Both are Odd}]}
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\( \mathbb{P}[\text{Sum is 10}] = \frac{3}{36} = \frac{1}{12} \).

\( \mathbb{P}[\text{Both are Odd}] = \frac{9}{36} = \frac{1}{4} \).
Two dice have both rolled odd. What are the chances the sum is 10?

\[ \mathbb{P}[\text{Sum is 10} \mid \text{Both are Odd}] = \frac{\mathbb{P}[(\text{Sum is 10}) \text{ AND } (\text{Both are Odd})]}{\mathbb{P}[\text{Both are Odd}]} \]

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3. \(P[(\text{Sum is 10}) \text{ AND (Both are Odd)}] = \frac{1}{36}\).
4. \(P[\text{Sum is 10} \mid \text{Both are Odd}] = \frac{1}{36} \div \frac{1}{4} = \frac{1}{9}\).

**Pop Quiz.** Compute \(P[\text{Both are Odd} \mid \text{Sum is 10}]\). Compare with \(P[\text{Sum is 10} \mid \text{Both are Odd}]\).
Computing a Conditional Probability

1: Identify that you need a conditional probability $\mathbb{P}[A \mid B]$.
2: Determine the probability space $(\Omega, \mathbb{P}(\cdot))$ using the outcome-tree method.
3: Identify the events $A$ and $B$ appearing in $\mathbb{P}[A \mid B]$ as subsets of $\Omega$.
4: Compute $\mathbb{P}[A \cap B]$ and $\mathbb{P}[B]$.
5: Compute $\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$. 
Best strategy is always switch. Winning outcomes: (2,3) or (3,2).

\[ P[\text{WinBySwitching}] = \frac{2}{3}. \]
Monty Prefers Door 3

Best strategy is always switch.
Winning outcomes: (2,3) or (3,2).
\[ \mathbb{P}[\text{WinBySwitching}] = \frac{2}{3}. \]

Perk up if Monty opens door 2!

- Intuition: Why didn’t Monty open door 3 if he prefers door 3?
Monty Prefers Door 3

<table>
<thead>
<tr>
<th>Prize</th>
<th>Host</th>
<th>Probability</th>
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<tbody>
<tr>
<td>1</td>
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<td>$P(1, 2) = \frac{1}{9}$</td>
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<tr>
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<td>3</td>
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$$P[\text{WinBySwitching}] = \frac{2}{3}.$$ 

Perk up if Monty opens door 2!

- Intuition: Why didn’t Monty open door 3 if he prefers door 3?

$$P[\text{Win}|\text{Monty opens Door 3}] = \frac{P[\text{Win AND Monty opens Door 3}]}{P[\text{Monty opens Door 3}]}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{9}}$$

$$= \frac{3}{4}.$$ 

Your chances improved from $\frac{2}{3}$ to $\frac{3}{4}$!
These four probabilities are all different.

\[
P[A] \quad P[A \mid B] \quad P[B \mid A] \quad P[A \text{ AND } B]
\]

Don’t use one when you should use another.
Conditional Probability Traps

These four probabilities are all different.

\[ P[A] \quad P[A \mid B] \quad P[B \mid A] \quad P[A \text{ AND } B] \]

Don’t use one when you should use another.

Sampling Bias: Using \( P[A] \) instead of \( P[A \mid B] \)

\[ P[\text{Voter will vote Republican}] \approx \frac{1}{2}. \]

Ask Apple™ to call up i-Phone™ users to see how they will vote.

\[ P[\text{Voter will vote Republican } \mid \text{Voter has an i-Phone}] \gg \frac{1}{2}. \quad \text{(Why?)} \]

This has trapped many US election-pollers. For a famous example, Google™ “Dewey Defeats Truman.”
Conditional Probability Traps

These four probabilities are all different.

\[ \mathbb{P}[A] \quad \mathbb{P}[A \mid B] \quad \mathbb{P}[B \mid A] \quad \mathbb{P}[A \text{ AND } B] \]

Don’t use one when you should use another.

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**Transposed Conditional: Using \( P[B \mid A] \) instead of \( P[A \mid B] \)**

Famous Lombard study on the riskiest profession: Student! Lombard confused:

\[ \mathbb{P}[\text{Student} \mid \text{Die Young}] \quad \text{with} \quad \mathbb{P}[\text{Die Young} \mid \text{Student}] \]
Two types of outcomes in any event $A$: 

\[ \Omega \]
Total Probability: Case by Case Probability

Two types of outcomes in any event $A$:
- The outcomes in $B$ (green);
Total Probability: Case by Case Probability

Two types of outcomes in any event $A$:
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Two types of outcomes in any event $A$:
- The outcomes in $B$ (green);
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$$P[A] = P[A \cap B] + P[A \cap \overline{B}] .$$  \hspace{1cm} (\ast) \\
(Similar to sum rule from counting.)
Total Probability: Case by Case Probability

Two types of outcomes in any event $A$:
- The outcomes in $B$ (green);
- The outcomes not in $B$ (red).

$$P[A] = P[A \cap B] + P[A \cap \overline{B}].$$  \hspace{1cm} (*)

(Similar to sum rule from counting.)

From the definition of conditional probability:

$$P[A \cap B] = P[A \text{ AND } B] = P[A | B] \times P[B];$$
$$P[A \cap \overline{B}] = P[A \text{ AND } \overline{B}] = P[A | \overline{B}] \times P[\overline{B}].$$
Total Probability: Case by Case Probability

Two types of outcomes in any event $A$:
- The outcomes in $B$ (green);
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$$P[A] = P[A \cap B] + P[A \cap \overline{B}].$$

(Similar to sum rule from counting.)

From the definition of conditional probability:

$$P[A \cap B] = P[A \text{ and } B] = P[A | B] \times P[B];$$
$$P[A \cap \overline{B}] = P[A \text{ and } \overline{B}] = P[A | \overline{B}] \times P[\overline{B}].$$

Plugging these into (*), we get a FUNDAMENTAL result for case by case analysis:

**Law of Total Probability**

$$P[A] = P[A | B] \cdot P[B] + P[A | \overline{B}] \cdot P[\overline{B}].$$

(Weight conditional probabilities for each case by probabilities of each case and add.)
Pick a random coin and flip. What is the probability of H?
Pick a random coin and flip. What is the probability of H?

**Outcome-Tree Method**

\[
P[H] = \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3}.
\]
Three Coins: Two Are Fair, One is 2-Headed

Pick a random coin and flip. What is the probability of H?

**Outcome-Tree Method**

![Outcome Tree Diagram]

**Total Probability**

Case 1. \(B\): You picked one of the fair coins
Case 2. \(\overline{B}\): You picked the two-headed coin

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Pick a random coin and flip. What is the probability of H?

**Outcome-Tree Method**

```
 biased  0 1/3
  
 fair 1/2 0 1/3
  
 fair 1/2 1/2 1/3
  
 H 1/2 1/2 1/3
```

**Total Probability**

Case 1. $B$: You picked one of the fair coins
Case 2. $\bar{B}$: You picked the two-headed coin

$$P[H] = P[H | B] \cdot P[B] + P[H | \bar{B}] \cdot P[\bar{B}]$$

$$= \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{2}{3} + \frac{1}{3} \cdot 1$$

$$= \frac{1}{12} + \frac{1}{9} + \frac{1}{3} = \frac{2}{3}.$$
Three Coins: Two Are Fair, One is 2-Headed

Pick a random coin and flip. What is the probability of H?

**Outcome-Tree Method**

<table>
<thead>
<tr>
<th>Event</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>biased</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>fair</td>
<td>$\frac{1}{3}$</td>
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<tr>
<td>fair</td>
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<tr>
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</tbody>
</table>

**Total Probability**

Case 1. $B$: You picked one of the fair coins
Case 2. $\overline{B}$: You picked the two-headed coin

\[
P[H] = P[H \mid B] \cdot P[B] + P[H \mid \overline{B}] \cdot P[\overline{B}]
\]
\[
= \frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}
\]
\[
= \frac{2}{3}.
\]
Three Coins: Two Are Fair, One is 2-Headed

Pick a random coin and flip. What is the probability of H?

**Outcome-Tree Method**

```
  H 1/3
 / \
H 1/3 0
 / \
T 1/6
 / \
T 1/6
```

**Total Probability**

\[
P[H] = P[H | B] \cdot P[B] + P[H | \overline{B}] \cdot P[\overline{B}]
\]

Case 1. \( B \): You picked one of the fair coins

Case 2. \( \overline{B} \): You picked the two-headed coin

\[
\begin{align*}
P[H] &= \frac{1}{2} \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} \\
&= \frac{2}{3}.
\end{align*}
\]

**Exercise.** A box has 10 coins: 6 fair and 4 biased (probability of heads \( \frac{2}{3} \)). What is \( P[2 \text{ heads}] \) in each case?

- Pick a single random coin and flip it 3 times.
- Flip 3 times. For each flip, pick a random coin, flip it and then put the coin back.
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Biased coin has *unknown* probability of heads $p$. Can you get a fair “toss”? 

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- If you get ‘ht’ output H; ‘th’ output T; otherwise RESTART.
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- This suggests that an H is as likely as a T.
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By the law of total probability (3 cases),

$$
P[H] = P[H | \text{RESTART}] \cdot P[\text{RESTART}] + P[H | ‘ht’] \cdot P[‘ht’] + P[H | ‘th’] \cdot P[‘th’]
$$

\[
\begin{array}{cccc}
P[H] & p^2 + (1 - p)^2 & 1 & p(1 - p) \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & p(1 - p) \\
\end{array}
\]
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By the law of total probability (3 cases),

$$
\mathbb{P}[H] = \mathbb{P}[H \mid \text{RESTART}] \cdot \mathbb{P}[\text{RESTART}] + \mathbb{P}[H \mid \text{‘ht’}] \cdot \mathbb{P}[\text{‘ht’}] + \mathbb{P}[H \mid \text{‘th’}] \cdot \mathbb{P}[\text{‘th’}]
$$

$$
= \mathbb{P}[H] \left( p^2 + (1 - p)^2 \right) + p(1 - p)
$$
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$$\mathbb{P}[H] = \mathbb{P}[H \mid \text{RESTART}] \cdot \mathbb{P}[\text{RESTART}] + \mathbb{P}[H \mid ‘ht’] \cdot \mathbb{P}[‘ht’] + \mathbb{P}[H \mid ‘th’] \cdot \mathbb{P}[‘th’]$$

$$\mathbb{P}[H] = p^2 + (1 − p)^2 \cdot 1 \cdot p(1 − p) \cdot 0 \cdot p(1 − p)$$

$$\mathbb{P}[H] = \mathbb{P}[H](p^2 + (1 − p)^2) + p(1 − p)$$

Solve for $\mathbb{P}[H]$

$$\mathbb{P}[H] = \frac{p(1 − p)}{1 − (p^2 + (1 − p)^2)} = \frac{p(1 − p)}{2p − 2p^2} = \frac{p(1 − p)}{2p(1 − p)} = \frac{1}{2}.$$