Overview

- This homework is due by 11:59:59 PM on Tuesday, March 6, 2018.
- This homework is to be completed **individually**. Do not share your work with anyone else.
- You **must** type your solutions for this homework assignment, then generate a PDF. Handwritten assignments will not be graded.
- You are strongly encouraged to use LaTeX, in particular for mathematical symbols. See references on the course website.
- Upload your PDF to Submitty; note there is a 1MB limit for file size.
- Please be concise and to the point in your answers. Even if your solution is correct, if it is not well-written, you may lose points.
- You have five late days to use throughout the semester. You may use at most three late days on this assignment.
- Please start your homework early and ask questions at office hours and your recitation section. Also ask (and answer) questions on Piazza.
- You can use either the new textbook or the textbook from last semester (i.e., F17). Problems are listed below from the current textbook; if the problem was numbered differently in the F17 textbook, the F17 problem is listed in square brackets.

Warm-up Exercises (before Wednesday Recitation)

1. **Problem 8.9(a)-(b) [or F17 Problem 8.8(a)-(b)]:** Recursively define rooted binary trees (RBT) and rooted full binary trees (RFBT).
   
   (a) Give examples, with derivations, of RBTs and RFBTs with 5, 6 and 7 nodes.
   (b) Prove by structural induction that every RFBT has an odd number of nodes.

2. **Problem 9.1(f)-(h) [this problem is not in the F17 textbook]:** Compute the following sums.

   (f) \[ \sum_{i=1}^{5} 2^i \]
   (g) \[ \sum_{i=1}^{5} (2^i)^2 \]
   (h) \[ \sum_{i=1}^{5} 2^{i^2} \]
3. **Problem 9.2(a)-(d) [or F17 Problem 9.1(a)-(d)]:** Tinker and then compute formulas that do not contain a sum for the following:

(a) \[ \sum_{i=1}^{n} 3i \]
(b) \[ \sum_{i=1}^{n} (3i + 2j) \]
(c) \[ 2n \sum_{i=1}^{n} (1 + 2^i) \]
(d) \[ \sum_{i=1}^{n} (3i + 2i^2) \]

4. **Problem 9.3(a)-(b) [or F17 Problem 9.2(a)-(b)]:** Compute formulas that do not contain a sum for the following:

(a) \[ \sum_{i=1}^{n} \sum_{j=1}^{m} (1 + j) \]
(b) \[ \sum_{i=1}^{n} \sum_{j=1}^{i} (1 + j) \]

5. **Problem 9.6(a) [or F17 Problem 9.4(a)]:** Compute formulas for:

(a) \[ \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} (i + j + k) \]

6. **Problem 9.12(a)-(f) [or F17 Problem 9.9(a)-(f)]:** Here are errors in the use of asymptotic notation. Explain why they are errors.

(a) \( 2n^2 + n = \Theta(n^2) \).
(b) \( 4^n \in \Theta(2^n) \) because 4 is a constant factor bigger than 2, and we ignore constants.
(c) \( O(1) + O(1) = O(1) \).
(d) Look! Your runtime \( T \in O(n^2) \), so your algorithm is slower than linear.
(e) Look! My runtime \( T \in o(n^3) \), so my algorithm is super fast (linear).
(f) \( f \in O(g) \) (i.e., “\( f \leq g \)”). Taking exponents on both sides, we conclude \( 2^f \in O(2^g) \).

7. **Problem 9.19(a)-(b) [or F17 Problem 9.16(a)-(b)]:** Prove or disprove:

(a) \[ \frac{n^3 + 2n}{n^2 + 1} \in \Theta(n) \]
(b) \( (n + 1)! \in \Theta(n!) \)

8. **Problem 9.51(a)-(b) [or F17 Problem 9.46(a)-(b)]:** See the textbook or the course website for the algorithm.

(a) Assume every operation (addition, assignment, comparison, \texttt{max}, etc.) takes 1 unit of time. Show that the running time \( T(n) \) is given by: \( T(n) = 5 + \sum_{i=1}^{n} 10 \).
(b) Show that \( T(n) = 5 + 10n \).
9. **Problem 10.2 [this problem is not in the F17 textbook]**: Kilam exercises every 12 days and Liamsi every 8 days. Kilam and Liamsi both exercised today. How many days until they exercise together again?

10. **Problem 10.3 [or F17 Problem 10.1]**: What natural numbers are relatively prime to 2, 3 and 6?

11. **Problem 10.6(a) [or F17 Problem 10.3(a)]**: Prove.
   (a) For \(m, n > 0\), \(\gcd(m, n) = \gcd(m, n - mx)\) for \(x \in \mathbb{Z}\).

12. **Problem 10.9 [or F17 Problem 10.5]**: Let \(m, n, d > 0\) and suppose \(d\) is a common divisor for \(m, n\), so \(d | m\) and \(d | n\). Suppose also that for some \(x, y \in \mathbb{Z}\), \(d = mx + ny\). Prove that \(d = \gcd(m, n)\).

13. **Problem 10.28(a)-(c) [or F17 Problem 10.24(a)-(c)]**: The hour hand is currently at 3. Where will the hour hand be after:
   (a) 233 hours?
   (b) \(14 \times 233\) hours?
   (c) \(233^{233}\) hours? (That’s a long time!)

14. **Problem 11.1**: Draw pictures of \(K_1, K_2, K_3, K_4, K_5, K_6\) and \(K_{4,4}\). Give the degree sequences of \(K_{n+1}, K_{n,n}, L_n, C_n, S_{n+1}\) and \(W_{n+1}\). (Use filled in circles for vertices.)

15. **Problem 11.4(a)-(d) [this problem is not in the F17 textbook]**: Give a graph satisfying the constraints or explain why it doesn’t exist.
   (a) The graph has 5 vertices each of degree 3.
   (b) The graph has 4 edges and vertices of degrees 1,2,3,4.
   (c) The graph has 4 vertices of degrees 1,2,3,4.
   (d) The graph has 6 vertices of degree 1,2,3,4,5,5.

**Recitation Exercises (before/during Wednesday Recitation)**

Note that there might not be time to cover all of these problems during recitation.

1. **Problem 8.13 [or F17 Problem 8.12]**: Prove that every RBT is connected, which means there is a way of using links to go from every node to every other node. (Structural induction.)

2. **Problem 8.28 [or F17 Problem 8.21]**: \(G(1) = 1; G(n) = G(n - 1)(1 - \frac{1}{n})\) for \(n > 1\). Prove \(G(n) = \frac{1}{n}\).
3. **Problem 9.1(p)-(r) [this problem is not in the F17 textbook]**: Compute the following sums.

   (p) \[ \sum_{i=1}^{3} \sum_{j=1}^{i} (i - j) \]

   (q) \[ \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} 2 \]

   (r) \[ \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} 2^{i+j+k} \]

4. **Problem 9.2(e)-(f) [or F17 Problem 9.1(e)-(f)]**: Tinker and then compute formulas that do not contain a sum for the following:

   (e) \[ \sum_{i=1}^{n} (i + 1)^2 \]

   (f) \[ \sum_{i=0}^{n} 2^{3+i} \]

5. **Problem 9.3(c)-(e) [or F17 Problem 9.2(c)-(e)]**: Compute formulas that do not contain a sum for the following:

   (c) \[ \sum_{i=1}^{n} \sum_{j=1}^{i} (1 + j)^2 \]

   (d) \[ \sum_{i=0}^{n} \sum_{j=0}^{m} 2^{i+j} \]

   (e) \[ \sum_{i=0}^{n} \sum_{j=0}^{i} 2^{i+j} \]

6. **Problem 9.6(b) [or F17 Problem 9.4(b)]**: Compute formulas for:

   (b) \[ \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} (i + j + k) \]

7. **Problem 9.23(a) [or F17 Problem 9.20(a)]**: Prove by contradiction:

   (a) \[ n^3 \not\in O(n^2) \]

8. **Problem 10.6(b) [or F17 Problem 10.3(b)]**: Prove.

   (b) If \( a \) divides \( bc \) and \( \gcd(a, b) = 1 \), then \( a \) divides \( c \).

9. **Problem 10.26(a)-(b) [this problem is not in the F17 textbook]**: Prove:

   (a) \( a \equiv b \pmod{d} \) and \( b \equiv c \pmod{d} \) \( \rightarrow \) \( a \equiv c \pmod{d} \).

   (b) \( a \equiv b \pmod{d} \) \( \rightarrow \) \( \gcd(a, d) = \gcd(b, d) \).

10. **Problem 10.27(a) [or F17 Problem 10.23[(a)]]**: Use modular arithmetic to solve these problems.

    (a) Compute the remainder when: (i) \( 2200^{2200} \) is divided by 3; (ii) \( 2014^{2014} \) is divided by 5.
Homework Problems (to be handed in and graded)

1. **[9 POINTS]** Problem 8.7(d)-(f) [or F17 Problem 8.6(d)-(f)]: The set $\mathcal{P}$ of parenthesis strings has a recursive definition as shown. (By default, nothing else is in $\mathcal{P}$, i.e., minimality.)

   1. $\varepsilon \in \mathcal{P}$
   2. $x \in \mathcal{P} \to [x] \in \mathcal{P}$
   3. $x, y \in \mathcal{P} \to xy \in \mathcal{P}$

   (d) Prove by structural induction that every string in $\mathcal{P}$ is balanced.
   (e) For a string $x \in \mathcal{P}$, define the imbalance as follows. Start on the left of $x$ and move right. Add +1 for every [ you encounter, and add −1 for every ] you encounter.
      
      (i) After you traverse $x$, what is the imbalance?
      (ii) Give an upper bound on the imbalance at any point in $x$.
      (iii) Prove by structural induction that at any point in $x$, imbalance $\geq 0$.
   (f) In the text we defined the set $\mathcal{M}$ of balanced and matched parentheses. Prove that $\mathcal{P} = \mathcal{M}$:
      
      (i) Prove by structural induction that every $x \in \mathcal{P}$ has a derivation using the rules for $\mathcal{M}$.
      (ii) Prove by structural induction that every $x \in \mathcal{M}$ has a derivation using the rules for $\mathcal{P}$.

2. **[10 POINTS]** Problem 8.20(a)-(b) [or F17 Problem 8.14(a)-(b)]: Recursively define the binary strings that contain more 0’s than 1’s.

   (a) Prove that every string in your set has more 0’s than 1’s.
   (b) Prove that every string which has more 0’s than 1’s is in your set.

3. **[6 POINTS]** Problem 9.2(g)-(h) [or F17 Problem 9.1(g)-(h)]: Tinker and then compute formulas that do not contain a sum for the following:

   (g) $\sum_{i=1}^{n} ij$
   (h) $\sum_{i=0}^{n} (i + j)^2$

4. **[9 POINTS]** Problem 9.3(f)-(h) [or F17 Problem 9.2(f)-(h)]: Compute formulas that do not contain a sum for the following:

   (f) $\sum_{i=0}^{n} \sum_{j=i}^{n} (1 + j)$
   (g) $\sum_{i=0}^{n} \sum_{j=0}^{i} 2^i$
   (h) $\sum_{i=0}^{n} \sum_{j=0}^{i} j2^i$
5. [5 POINTS] Problem 9.15(a)-(e) [or F17 Problem 9.12(a)-(e)]:
   (a) 10
   (b) 3n + 9
   (c) $\lceil n \rceil$
   (d) $\lfloor n/2 \rfloor$
   (e) $n^2 + n + 1$

6. [10 POINTS] Problem 9.23(b) [or F17 Problem 9.20(b)]: Prove by contradiction:
   (b) $2^n \notin \Theta(3^n)$

7. [14 POINTS] Problem 9.48(a)-(b) [or F17 Problem 9.43(a)-(b)]: See the textbook or the course website for the algorithm.
   (a) Assume every operation (addition, assignment, comparison, max, etc.) takes 1 unit of time. Show that the running time $T(n)$ is given by:

   $$T(n) = 2 + \sum_{i=1}^{n} \left[ 2 + \sum_{j=i}^{n} \left( 5 + \sum_{k=i}^{j} 4 \right) \right]$$

   (b) Evaluate the triple nested-sum to show that $T(n) = 2 + \frac{35}{6}n + \frac{9}{2}n^2 + \frac{2}{3}n^3$.

8. [8 POINTS] Problem 10.4(a) [or F17 Problem 10.2(a)]: Use Euclid’s algorithm and the remainders generated to solve these problems.
   (a) Compute gcd(2250, 1200) and find $x, y \in \mathbb{Z}$ for which gcd(2250, 1200) = 2250$x + 1200y$.

9. [8 POINTS] Problem 10.8(a) [or F17 Problem 10.4(a)]: You may find Bezout’s identity useful for answering these questions.
   (a) Prove that consecutive integers $n$ and $n + 1$ are relatively prime.

10. [8 POINTS] Problem 10.27(b) [or F17 Problem 10.23[(b)]: Use modular arithmetic to solve these problems.
   (b) What is the last digit of $3^{2016} + 4^{2016} + 7^{2016}$?

11. [7 POINTS] Problem 11.3: A graph is regular if every vertex has the same degree. Which of these graphs are regular: $K_6$; $K_{4,5}$; $K_{5,5}$; $L_6$; $S_6$; $W_4$; $W_5$?

12. [6 POINTS] Problem 11.8(a)-(c) [or F17 Problem 11.7(a)-(c)]: Compute the number of edges in the following graphs: (a) $K_n$; (b) $K_{n,t}$; (c) $W_n$.

Submission Instructions

To submit your assignment your code, as noted on page 1, please generate a PDF of typewritten work, then submit the PDF via Submitty, the homework submission server. The specific URL is on the course website.

Be sure you submit only your PDF file.

And only submit solutions to problems posed in the Homework Problems section above.