1. **[4 POINTS]** The hour hand on a standard 12-hour clock is currently at 5:00. Where will the hour hand be after 97^(99) hours?

   **ANSWER:** 6:00

   First note that 96 divides 12. Then, 97 ≡ 1 (mod 12). From this, 97^99 ≡ 1^99 (mod 12). And 97^99 ≡ 1 (mod 12). Therefore, one hour after 5:00.

2. **[4 POINTS]** What is the last digit of 7^2018?

   **ANSWER:** 9

   First observe by tinkering the following pattern:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>7^n</td>
<td>7</td>
<td>49</td>
<td>343</td>
<td>2401</td>
<td>16807</td>
<td>117649</td>
<td>823543</td>
<td>5764801</td>
<td>...</td>
</tr>
<tr>
<td>last digit</td>
<td>7</td>
<td>9</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>9</td>
<td>3</td>
<td>1</td>
<td>...</td>
</tr>
</tbody>
</table>

   When n is a multiple of 4, the last digit is 1. Since 2016 is a multiple of 4, we move forward two positions to identify 9 as the last digit for 7^2018.

3. **[4 POINTS]** Give a formula for the sum \( S(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} (2i - j) \).

   **ANSWER:** \( S(n) = \frac{1}{2} n^2 (n + 1) \)

   \[
   S(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} 2i - \sum_{i=1}^{n} \sum_{j=1}^{n} j
   = 2n \sum_{i=1}^{n} i - \sum_{i=1}^{n} \frac{1}{2} n(n + 1)
   = 2n \frac{1}{2} n(n + 1) - \frac{1}{2} n^2 (n + 1)
   = n^2 (n + 1) - \frac{1}{2} n^2 (n + 1)
   = \frac{1}{2} n^2 (n + 1)
   \]
4. **[4 POINTS]** Given \( f(n) \) is a function that satisfies the recurrence \( f(n) = f(n-1) + \sqrt{n} \) with \( f(0) = 8 \), which order relationship describes \( f \)?

**ANSWER:** \( f \in \Theta(n \sqrt{n}) \)

Unravel the recursion:
\[
\begin{align*}
f(n) &= f(n-1) + \sqrt{n} \\
f(n-1) &= f(n-2) + \sqrt{n-1} \\
f(n-2) &= f(n-1) + \sqrt{n-2} \\
& \quad \vdots \\
f(1) &= f(0) + \sqrt{1}
\end{align*}
\]

From this pattern, we can write: \( f(n) = f(0) + \sum_{j=1}^{n} \sqrt{j} \).

And then, eliminating constant \( f(0) \), we have \( f(n) \in \Theta(n \sqrt{n}) \).

5. **[4 POINTS]** How many rows in the truth table for \((p \to q) \land (q \leftrightarrow \neg p)\) are \( F \)?

**ANSWER:** 3; the truth table is as follows:

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \to q )</th>
<th>( \neg p )</th>
<th>( q \leftrightarrow \neg p )</th>
<th>( (p \to q) \land (q \leftrightarrow \neg p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
</tbody>
</table>

6. **[4 POINTS]** Which statement is not true?

**ANSWER:** \((A \cup B) \cap \overline{A} = B \cup \overline{A}\)

Can verify the other four options are true (i.e., process of elimination).

Or use a Venn diagram to determine that \((A \cup B) \cap \overline{A} \neq B \cup \overline{A}\)

7. **[4 POINTS]** Given \( P(1) \) is \( T \) and \( P(n) \) is a predicate with \( P(n) \to (P(3n) \land P(5n)) \) for \( n \geq 1 \), which of the following statements is true?

**ANSWER:** \( P(n) \) is \( T \) for all positive powers of 3 and 5

Observe that \( P(1) \to P(3) \land P(5) \), and that \( P(3) \to P(9) \land P(15) \) and \( P(5) \to P(15) \land P(25) \).

Keep going to observe that all positive powers of 3 and all positive powers of 5 are \( T \).

Note that other values are also \( T \) (e.g., \( P(15) \), \( P(45) \), \( P(75) \), etc.).
8. [4 POINTS] Given set $S$ defined recursively as follows:

1. $2 \in S$
2. $x \in S \Rightarrow x^2 \in S$
   
   $x, y \in S \Rightarrow x^2 + y^2 \in S$

Which of the following are elements of $S$?

**ANSWER:** 2, 8, 20, 404

$2 \in S$

$2 \in S \Rightarrow 2^2 + 2^2 \in S \Rightarrow 8 \in S$

$2 \in S \Rightarrow 2^2 \in S \Rightarrow 2^2 + 4^2 \in S \Rightarrow 20 \in S$

$2 \in S \Rightarrow 2^2 \in S \Rightarrow 2^2 + 4^2 \in S \Rightarrow 2^2 + 20^2 \in S \Rightarrow 404 \in S$

9. [4 POINTS] Given the following recursive function:

$$f(n) = \begin{cases} 
2, & n = 1 \\
2n \times f(n - 1), & n > 1
\end{cases}$$

Which of the following functions is equivalent?

**ANSWER:** $f(n) = 2^n \times n!$

Observe the first few values from the recursive function:

$f(2) = 2 \times 2 \times f(1) = 2 \times 2 \times 2$

$f(3) = 2 \times 3 \times f(2) = 2 \times 3 \times 2 \times 2 \times 2$

Rearrange based on the emerging pattern:

$f(n) = (2 \times 2 \times \ldots \times 2)(n \times (n - 1) \times \ldots \times 2 \times 1)$

$f(n) = 2^n \times n!$

10. [4 POINTS] Given set $Q$ defined recursively as follows:

1. $\varepsilon \in Q$
   
   $0 \in Q$
   
   $1 \in Q$
2. $x \in Q \Rightarrow 0 \bullet x \bullet 0 \in Q$
   
   $x \in Q \Rightarrow 1 \bullet x \bullet 1 \in Q$

How many strings of length 8 can be generated?

**ANSWER:** 16

Given that the recursive definition of $Q$ is all binary string palindromes, then each member of set $Q$ is $zz^R$ or $z0z^R$ or $z1z^R$.

For strings of length 8, we use only $zz^R$ since the other two forms are of odd length.

Then, $z$ is of length 4; and there are $2^4 = 16$ binary strings of length 4.
11. [16 POINTS] Answer the questions below regarding sums and asymptotics.

(a) (6 points) Compute a formula that does not contain a sum for the following:

\[ f(n) = \sum_{j=0}^{n} (-1)^j j \]

**ANSWER:** \( f(n) = (-1)^n \lfloor \frac{n}{2} \rfloor \)

Expand a few terms to see the pattern:

\[ f(n) = 0 - 1 + 2 - 3 + 4 - 5 + 6 - 7 + \ldots \]

If \( n \) is odd, then each successive pair of terms sums to \(-1\).

In the odd case, \( f(n) = -\frac{n+1}{2} \) or \( f(n) = -\lfloor \frac{n}{2} \rfloor \). [1 point for showing odd case]

If \( n \) is even, then, excluding the initial 0 term, each successive pair of terms sums to 1.

In the even case, \( f(n) = \frac{n}{2} \). [1 point for showing even case]

A concise way to combine these two cases is: \( f(n) = (-1)^n \lfloor \frac{n}{2} \rfloor \). [4 points for concise solution]

[Less concise ways of writing this are worth 2 or 3 points instead of 4.]

(b) (4 points) Show that \( \frac{1}{n} + \frac{11}{n^2} \in \Theta(\frac{1}{n}) \).

**ANSWER:** [3 points for the work below]

\[
\lim_{n \to \infty} \left( \frac{n+11}{n^2} \right) = \lim_{n \to \infty} \left( \frac{n+11}{n^2} \right) \left( \frac{n}{n} \right) = \lim_{n \to \infty} \frac{n+11}{n} = 1
\]

[1 point for concluding as follows] Given that the above limit yields a constant greater than zero (i.e., 1), we have shown that \( \frac{1}{n} + \frac{11}{n^2} \in \Theta(\frac{1}{n}) \).

(c) (6 points) Given \( f(n) = \sum_{i=1}^{n} (2i)^2 \), describe how \( f(n) \) is asymptotically related to \( n, n^2 \), and \( n^3 \). More specifically, use \( o() \), \( O() \), \( \Theta() \), \( \Omega() \), and/or \( \omega() \) in relation to \( n, n^2 \), and \( n^3 \). Be as precise as possible.

**ANSWER:** \( f(n) \in \Theta(n^3) \); \( f(n) \in \omega(n^2) \); \( f(n) \in \omega(n) \) [2 points each; no partial credit]

\[
\begin{align*}
    f(n) &= \sum_{i=1}^{n} (2i)^2 \\
    f(n) &= 4 \sum_{i=1}^{n} i^2 \\
    f(n) &= 4 \times \frac{1}{6} n(n+1)(2n+1) \\
    f(n) &= \frac{2}{3} (n^2 + n)(2n + 1) \\
    f(n) &= \frac{2}{3} (2n^3 + 3n^2 + n)
\end{align*}
\]
12. **[15 POINTS]** Use induction to prove that \(n^2 - 1\) is divisible by 8 for all odd natural numbers \(n \geq 1\).

**GRADING RUBRIC:**

- [7 points for setting up the induction correctly]
- [4 points for getting most of the way through the proof correctly]
- [4 points for completing the induction proof correctly]

**ANSWER:** [note that other possible approaches may work here]

Using leaping induction, we prove claim \(P(n)\) that \(n^2 - 1\) is divisible by 8 for all odd natural numbers \(n \geq 1\).

**Base case.** With \(n = 1\), we have \(1^2 - 1 = 0\), which is divisible by 8.

**Induction step.** Assume \(P(n)\) is \(T\), i.e., that \(n^2 - 1\) is divisible by 8 for all odd natural numbers \(n \geq 1\).

Since we are looking at odd numbers, we must show that \(P(n + 2)\) is \(T\), i.e., that \((n + 2)^2 - 1\) is divisible by 8.

Considering the \(n + 2\) case, we have:

\[(n + 2)^2 - 1 = n^2 + 4n + 3 = (n^2 - 1) + 4n + 4 = (n^2 - 1) + 4(n + 1)\]

Using the inductive hypothesis, we note that \((n^2 - 1)\) is divisible by 8; therefore, we must show that \(4(n + 1)\) is also divisible by 8.

Since \(n\) is odd, we can rewrite as \(n = 2k + 1\).

Plugging this in, we have \(4(n + 1) = 4(2k + 1 + 1) = 4(2k + 2) = 8(k + 1)\), which is clearly divisible by 8.

From the above, we have shown that \(P(n + 2)\) is \(T\), thereby completing our inductive proof.
13. **(15 POINTS)** Use induction to prove that $n! > n^3$ for all natural numbers $n \geq 6$.

**GRADING RUBRIC:**

- [7 points for setting up the induction correctly]
- [4 points for getting most of the way through the proof correctly]
- [4 points for completing the induction proof correctly]

**ANSWER:** [note that other possible approaches may work here]

Using induction, we prove claim $P(n)$ that $n! > n^3$ for all natural numbers $n \geq 6$.

**Base case.** With $n = 6$, we have $6! > 6^3$, or $720 > 216$.

**Induction step.** Assume $P(n)$ is $T$, i.e., that $n! > n^3$ for all natural numbers $n \geq 6$.

We must show that $P(n + 1)$ is $T$, i.e., that $(n + 1)! > (n + 1)^3$ is $T$.

Considering the $n + 1$ case, we have $(n + 1)! = n!(n + 1)$.

Using the inductive hypothesis, we then have $n!(n + 1) > n^3(n + 1) = n^4 + n^3$.

Since we are trying to get to $(n + 1)^3$, we can expand as follows:

$$(n + 1)^3 = (n + 1)(n^2 + 2n + 1)$$

$$= n^3 + 2n^2 + n + n^2 + 2n + 1$$

$$= n^3 + 3n^2 + 3n + 1$$

Combining the above, we need to show that $(n + 1)! > n^4 + n^3$, and since we already have an $n^3$ term, we must prove claim $Q(n)$: $n^4 > 3n^2 + 3n + 1$ is $T$ for $n \geq 6$.

We again use induction here, starting with a base case of $n = 6$

in which $6^4 > 3(6^2) + 3(6) + 1$ or $1296 > 127$.

We assume that $Q(n)$ is $T$ for $n \geq 6$ and must show that $Q(n + 1)$ is $T$,

i.e., that $(n + 1)^4 > 3(n + 1)^2 + 3(n + 1) + 1$ is $T$.

Considering the $n + 1$ case, we have

$$(n + 1)^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 > 3(n + 1)^2 + 3(n + 1) + 1 = 3n^2 + 9n + 7, \text{ which is } T.$$ 

From the above, since we have shown that $Q(n + 1)$ is $T$, we have shown that $P(n + 1)$ is $T$, thereby completing our inductive proof.
14. **[14 POINTS]** Answer the questions below regarding recursively defined sets and structural induction.

(a) **(5 points)** Give a recursive definition for the set $\mathcal{B}$ that contains all binary strings of even length.

**ANSWER:** [note that other possible approaches may work here]

Set $\mathcal{B}$ is defined recursively as follows:

1. $\varepsilon \in \mathcal{B}$
2. $x \in \mathcal{B} \rightarrow x \cdot 00 \in \mathcal{B}$
   
   $x \in \mathcal{B} \rightarrow x \cdot 01 \in \mathcal{B}$
   
   $x \in \mathcal{B} \rightarrow x \cdot 10 \in \mathcal{B}$
   
   $x \in \mathcal{B} \rightarrow x \cdot 11 \in \mathcal{B}$

Or as:

1. $\varepsilon \in \mathcal{B}$
2. $x \in \mathcal{B} \rightarrow 0 \cdot x \cdot 0 \in \mathcal{B}$
   
   $x \in \mathcal{B} \rightarrow 0 \cdot x \cdot 1 \in \mathcal{B}$
   
   $x \in \mathcal{B} \rightarrow 1 \cdot x \cdot 0 \in \mathcal{B}$
   
   $x \in \mathcal{B} \rightarrow 1 \cdot x \cdot 1 \in \mathcal{B}$

(b) **(9 points)** Use structural induction to prove that every string in $\mathcal{B}$ is a binary string of even length.

**GRADING RUBRIC:**

- [4 points for setting up the induction correctly]
- [2 points for getting most of the way through the proof correctly]
- [3 points for completing the induction proof correctly]

**ANSWER:** [note this may vary depending on the answer to (a)]

Given set $\mathcal{B}$ defined in (a) above, we also define property $\mathcal{P}(s)$ for any element $s \in \mathcal{B}$ to be that $s$ is a binary string of even length.

**Base case.** We must show that for each basis $s_i$ of (a), $\mathcal{P}(s_i)$ is $T$.

In this case, $\mathcal{P}(\varepsilon)$ is $T$ since a length of zero is even.

**Inductive step.** We must show for each and every constructor rule that if $\mathcal{P}$ is $T$ for the parent element(s), then $\mathcal{P}$ is $T$ for the new child element that is constructed.

For each constructor rule from (a), exactly two binary digits are added to the new child element. Therefore, the length increases by 2, ensuring it remains even.

By structural induction, we conclude that $\mathcal{P}(s)$ is $T$ for all $s \in \mathcal{P}$.
15. [+2 POINTS EXTRA CREDIT] What is the month and day of Professor Goldschmidt’s son’s birthday (as mentioned in class on March 5)?

[+1 point] April 1

And if he was born in 2010, what day of the week was he born on?

[+1 point] Thursday (this is a modular arithmetic problem)