1. **[6 POINTS]** For language $L_1 = \{0^n1^m \mid n, m \geq 1, m \geq n\}$, which string is in $L_1$?

**ANSWER:** $001111$ is in $L_1$ (with $n = 3, m = 4$).

Note that the other strings ($0111110, 0000000, 0000111$) are not in $L_1$.

2. **[6 POINTS]** Let $A$ and $B$ both be countable sets. The Cartesian product $A \times B$ is the set of 2-tuples $(a, b)$, where $a \in A$ and $b \in B$. What can be said about the claim that $A \times B$ is countable?

**ANSWER:** The claim is true. Since $A$ and $B$ are both countable sets, taking elements $a \in A$ and $b \in B$ to form 2-tuples $(a, b)$ is also countable.

If $A$ and $B$ are both finite, then the resulting Cartesian product is certainly countable (since any finite set is countable).

If $A$ and/or $B$ are infinite, write out the Cartesian product $A \times B$ as $(a_0, b_0), (a_0, b_1), (a_0, b_2), \ldots, (a_1, b_0), (a_1, b_1), (a_1, b_2), \ldots$, and so on. Though infinite, we can list all elements and therefore map them to $\mathbb{N}$.

As a more formal approach, we observe that since $A$ and $B$ are countable, there is an injection from $A$ to $\mathbb{N}$ and an injection from $B$ to $\mathbb{N}$. It follows that there is an injection from $A \times B$ to $\mathbb{N}^2$ and that $|\mathbb{N}^2| = |\mathbb{N}|$.

3. **[6 POINTS]** Which of the following sets has the same cardinality as $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$?

**ANSWER:** First note that $\mathbb{N}_0$ is an infinite set.

Given $J = \{1, 2, 3, 4, \ldots, 100\}$, $G = \{\text{the set of all connected graphs with edge set } E \text{ and with } |E| \leq 100\}$, and $M = \{M_1, M_2, M_3, \ldots\}$ (the set of all possible Turing Machines).

Since sets $J$ and $G$ are finite sets, they do not have the same cardinality as $\mathbb{N}_0$. The set of all Turing Machines $M$ is an infinite set that can be mapped to $\mathbb{N}_0$. Therefore, the answer is $M$.

4. **[6 POINTS]** Which of the following strings, if any, is not in the language $L_4$ described by the regular expression $\{1, 010\}^*$?

**ANSWER:** $11011010$ is not in the language $L_4$ because we cannot construct this string from zero or more occurrences of 1 and 010.

Note that we can construct strings $10101010$, $\varepsilon$, and $11111111$ from zero or more occurrences of 1 and 010.
5. [6 POINTS] Given $|Z| = |N|$ and the list-position $\ell_z$ of element $z \in Z$ defined as:

$$
\ell_z = \begin{cases} 
2z & \text{if } z > 0 \\
2|z| + 1 & \text{if } z \leq 0
\end{cases}
$$

What is the list-position $\ell_z$ of element $z = -73$?

**ANSWER:** List-position $\ell_z$ for element $z = -73$ is $2|z| + 1 = 2| -73 | + 1 = 147$.

6. [6 POINTS] Given the following language:

$$
L_6 = \{0100, 011000, 001000, 00110000, 00010000, 01110000, 0011100000, \ldots\}
$$

Which of the following best describes this language?

**ANSWER:** $L_6 = \{0^m1^m0^{(m+n)} \mid m, n > 0\}$

7. [6 POINTS] For the DFA below, what is an equivalent regular expression?

![DFA Diagram](image-url)

**ANSWER:** All of the given answers are correct. More specifically, the following regular expressions accept strings that the given DFA accepts: $\{0,1\}^*11^*; \ast 1; 0^*\{10^+\}^*1$.

8. [6 POINTS] Which computing problem below, if any, cannot be solved by a CFG? Assume alphabet $\Sigma = \{0, 1\}$.

**ANSWER:** There were actually two correct answers to this question.
Both $L_8 = \{w01\#10w \mid w = \Sigma^*\}$ and $L_8 = \{1^m0^*2^n1^m \mid n > 0\}$ cannot be solved by a CFG.
Further, $L_8 = \{\text{all strings with an even number of 0’s and an odd number of 1’s}\}$ and $L_8 = \{\text{all strings with at least five 1’s}\}$ can be solved using a DFA (and therefore a CFG).

9. [6 POINTS] What is the computing problem $L_9$ solved by the CFG below? In other words, what is the language generated by the CFG? Assume alphabet $\Sigma = \{0, 1\}$.

$$
S \rightarrow \varepsilon \mid 0S1 \mid 1S1
$$

**ANSWER:** $L_9 = \{u \cdot v \mid u \in \Sigma^* \text{ and } v = 1^{\cdot |u|}\}$
10. **[6 POINTS]** What is the language $L_{10}$ recognized by Turing Machine $M_{10}$ below? In other words, for what input(s) does $L_{10}(M_{10})$ halt in accept state $A$?

**ANSWER:** $L_{10}(M_{10}) = \{1^*0, 1^*00, 1^*0#\}$

11. **[6 POINTS]** A random binary string $b_1b_2b_3 \cdots b_7$ of length 7 is input into the DFA below, with $\Pr[b_i \text{ is even}] = 0.5$. What is the probability that the string is accepted?

**ANSWER:** $\frac{29}{128}$

The DFA accepts strings containing zero, one, or two 1’s. For strings of length 7, there is one string containing zero 1’s (i.e., 0000000), $\binom{7}{1} = 7$ strings containing one 1, and $\binom{7}{2} = 21$ strings containing two 1’s.

Of the $2^7 = 128$ possible bit strings, $1 + 7 + 21 = 29$ strings that the DFA would accept.
12. [6 POINTS] The computing problem $L_{12} = \{ \text{all strings of length at least 3 with an odd number of 0's} \}$ can be solved by which of the following?  

(I) DFA   (II) CFG   (III) Turing Machine  

**ANSWER:** A DFA can solve this computing problem. Since this can be solved by a DFA, then it can also be solved by a CFG and a Turing Machine.

13. [6 POINTS] The computing problem $L_{13} = \{ 0^{3i}10^{3j}10^{3k} \mid i, j, k \geq 0, k = i + j \}$ can be solved by which of the following?  

(I) DFA   (II) CFG   (III) Turing Machine  

**ANSWER:** Given the need for memory, a DFA cannot solve this computing problem. We can use the following CFG to solve this problem:  

$$  
S \rightarrow 000S000 \mid A \\
A \rightarrow 1B \\
B \rightarrow 000B000 \mid 1 
$$  

Therefore, a Turing Machine can also solve this computing problem.

14. [6 POINTS] Which of the following are countable?  

**ANSWER:** All three of these sets are countable.  

$\mathbb{Z}$ is countable as shown in Chapter 22.  
The set of all odd numbers in $\mathbb{N}$ are countable, similar to the approach shown in Chapter 22 for all even numbers.  
The set of all prime numbers is countable since it forms an injection to $\mathbb{N}$.

15. [8 POINTS] For given language $L_{15} = \{ 0^n1^m0^{2n} \mid n, m \geq 1, m \geq n \}$, how many strings of length $\ell \leq 8$ are in $L_{15}$?  

**ANSWER:** Exactly 6 strings of length $\ell \leq 8$ are in $L_{15}$:  

$n = 1, 2n = 2, m = 1 \implies 0100$  
$n = 1, 2n = 2, m = 2 \implies 01100$  
$n = 1, 2n = 2, m = 3 \implies 011100$  
$n = 1, 2n = 2, m = 4 \implies 0111100$  
$n = 1, 2n = 2, m = 5 \implies 01111100$  
$n = 2, 2n = 4, m = 2 \implies 00110000$
16. [8 POINTS] Given the CFG defined as:

\[
\begin{align*}
S & \rightarrow P \mid S + P \\
P & \rightarrow T \mid P \times T \\
T & \rightarrow 8 \mid (S)
\end{align*}
\]

How many different derivations are there of string \(8 + 8 \times (8 + 8)\)?

**Answer:** There are many derivations (far more than 4) of string \(8 + 8 \times (8 + 8)\).

One derivation is:

\[
\begin{align*}
S & \rightarrow S + P \rightarrow S + P \times T \rightarrow S + T \times T \rightarrow S + T \times (S) \\
& \rightarrow S + T \times (S + P) \rightarrow S + T \times (S + T) \rightarrow P + T \times (S + T) \rightarrow T + T \times (S + T) \\
& \rightarrow T + T \times (P + T) \rightarrow T + T \times (T + T) \rightarrow 8 + T \times (T + T) \rightarrow 8 + 8 \times (T + T) \\
& \rightarrow 8 + 8 \times (8 + T) \rightarrow 8 + 8 \times (8 + 8)
\end{align*}
\]

Note that only one variable (i.e., \(S, P, T\)) is replaced at each step. Change the order of replacements to form a different derivation.