

# Uncertainty and Probability

# Using probability

- The general idea is to assign a probability or *confidence factor* to:
  - each fact in a database.
  - decision in a game
- When using the database to answer questions - we find the *most probable answer*.

# Basic Probability Concepts

- $P(A)$  is the probability that  $A$  is true.
  - *Unconditional* probability - does not depend on any other factors. a.k.a *prior* probability.
  - $P(\neg A) = 1 - P(A)$
- $P(A \vee B)$  is the unconditional probability that either  $A$  or  $B$  is true.

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

# Joint Probabilities

- The collection of the probabilities for the possible values of all statements defines the joint probabilities. Example:

	Toothache	$\neg$ Toothache
Cavity	0.04	0.06
$\neg$ Cavity	0.01	0.89

# Conditional Probability

$P(A|B)$  is the conditional probability that  $A$  is true assuming  $B$  is true, and  $B$  is everything we know (that is relevant).

$$P(A|B) = P(A \wedge B) / P(B)$$

NOTE:  $P(A|B) P(B) = P(B|A) P(A)$

# Bayes Rule

- Bayes rule gives an alternative formula for computing conditional probability:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- This allows us to compute conditional probability when we don't know  $P(A \wedge B)$ .

# Sick Example

- The probability that measles causes spots is 0.5.
- The probability that a person has measles is 0.1.
- The probability that a person has spots is 0.2.
- What is the probability that a person who has spots has measles ?

# Spots & Measles

$$P(\text{Spots}|\text{Measles}) = 0.5$$

$$P(\text{Measles}) = 0.1$$

$$P(\text{Spots}) = 0.2$$

$$P(\text{Measles}|\text{Spots}) = \frac{P(\text{Spots}|\text{Measles})P(\text{Measles})}{P(\text{Spots})}$$

$$P(\text{Measles}|\text{Spots}) = \frac{(0.5)(0.1)}{0.2} = 0.25$$

# Relative Probabilities

If we know that  $P(A|B_1) = P(A|B_2)$ , then

$$\frac{P(B_1|A)}{P(B_2|A)} = \frac{P(B_1)}{P(B_2)}$$

Since knowing about  $B_1$  tells us just as much about  $A$  as knowing about  $B_2$ ,

knowing about  $A$  doesn't change the relative probabilities of  $B_1$  and  $B_2$

# Example

- The probability traffic light T is
  - Green is 0.45
  - Yellow is 0.1
  - Red is 0.45
- We also know that anyone who runs a red light gets a ticket (always) and nobody else gets a ticket.
- We crossed without getting a ticket - what is the probability the light was yellow? Green? Red?

# Example (cont.)

$P(\text{Red})=0.0$  (since we didn't get a ticket)

$P(\text{No-Ticket} \mid \text{Green}) = P(\text{No-Ticket} \mid \text{Yellow})$

So we know:

$$\frac{P(\text{Yellow} \mid \text{No-Ticket})}{P(\text{Green} \mid \text{No-Ticket})} = \frac{P(\text{Yellow})}{P(\text{Green})} = \frac{0.1}{0.45}$$

we also know that

$$P(\text{Yellow} \mid \text{No-Ticket}) + P(\text{Green} \mid \text{No-Ticket}) = 1.0$$

$$P(\text{Yellow} \mid \text{No-Ticket}) = (.1/.55)$$

$$P(\text{Green} \mid \text{No-Ticket}) = (.45/.55)$$

# Shell Game / Monty Hall Problem (famous problem).

- A pea is placed under 1 of 3 shells, your job is to pick the right shell.
- Once you pick a shell, the game operator turns over a shell you did not pick and that does not cover the pea. He/she now allows you to change your selection.
- Should you change your selection?

# Bayesian Analysis

- Shells are A, B and C.
- Each proposition A, B and C is true only if the pea is under that shell.
- Assume you pick shell A (we call A the one you pick).
- We want to know the probability you will win if you stick with A, and the probability you will win if you switch to another shell

# Stick with your choice

- The probability that you win if you don't change is:

$$P(A|\neg B \vee \neg C) = \frac{P(\neg B \vee \neg C|A)P(A)}{P(\neg B \vee \neg C)}$$

$$P(A|\neg B \vee \neg C) = \frac{P(\neg B \vee \neg C|A)P(A)}{P(\neg B \vee \neg C)}$$

- $P(\neg B \vee \neg C | A)$  is 1.0, since if the pea is under A we know it is not under B or C.
- $P(A) = 1/3$
- $P(\neg B \vee \neg C)$  is 1.0, since the pea can't be under both B and C.
- $P(A | \neg B \vee \neg C) = 1/3$ . This is to be expected since turning over an empty shell does not change our original selection.

# Switch Shells

- The probability you win if you do switch:

$$P(\neg A | \neg B \vee \neg C) = \frac{P(\neg B \vee \neg C | \neg A) P(\neg A)}{P(\neg B \vee \neg C)}$$

$$P(\neg A | \neg B \vee \neg C) = \frac{P(\neg B \vee \neg C | \neg A) P(\neg A)}{P(\neg B \vee \neg C)}$$

- $P(\neg B \vee \neg C | \neg A)$  is 1, since the pea can't be under both B and C.
- $P(\neg A) = 2/3$
- $P(\neg B \vee \neg C)$  is 1.0, since the pea can't be under both B and C.
- $P(\neg A | \neg B \vee \neg C) = 2/3$ . You should switch!

# Another analysis

## List all possibilities

			Stay	Switch
A	$\neg$ B	$\neg$ C	win	lose
$\neg$ A	B	$\neg$ C	lose	win
$\neg$ A	$\neg$ B	C	lose	win

# Shell Game Intuition

- If you stick with your original choice, you will obviously win  $1/3$  of the time.
- If you switch, then unless your original choice was correct you will win. This happens  $2/3$  of the time.

# Exercise- are you doomed?

- The doctor says you have tested positive for the disease *lecturitis sleepitosis maximitus*.
- This disease is always fatal within 24 hours.
- The test is 99% accurate
  - if you have the disease the test will say so 99% of the time.
  - if you don't have the disease the test will say so 99% of the time.
- This a rare disease, only 1 in 10,000 people have it.