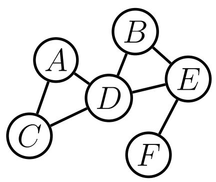
# Foundations of Computer Science
# Lecture 3

## Making Precise Statements

Propositions
Compound Propositions and Truth Tables
Predicates and Quantifiers

# Last Time

1. Sets, $\{3, 5, 11\}$
2. Sequences, 100111001
3. Graphs,




4. Examples of basic proofs.
   - In 4 rounds of group dating, no one meets more than 12 people.
   - $x^2$ is even "is the same as" $x$ is even.
   - In *any* group of 6 people there is an orgy of 3 mutual friends or a war of 3 mutual enemies.
   - **Axiom:** The Well-Ordering Principle
   - $\sqrt{2}$ is not rational.

# Today: Making Precise Statements

1. Making a precise statement: the proposition
2. Complicated precise statements: the compound proposition
   - Truth tables
3. Claims about many things
   - Predicates
   - Quantifiers
   - Proofs with quantifiers

# Statements can be Ambiguous

1. 2+2=4. T
2. 2+2=5. F
3. You may have cake <u>OR</u> ice-cream. (Can you have both?)
4. <u>IF</u> pigs can fly <u>THEN</u> you get an $A$. (Pigs can't fly. So, can you get an A?)
5. <u>There is</u> one soulmate for <u>EVERY</u> person.
   - 5(a) There is a single soul mate that <u>EVERY</u> person shares.
   - 5(b) <u>EVERY</u> person has their own special soul mate.

Why is ambiguity bad? **Proof!**

We asked questions of our friends to prove 5(b).

$A$ says Sue's their soul mate;
$B$ says Joe's their soul mate;
$C$ says Sue's their soul mate;
$D$'s soul mate is a red Porshe;
$E$ says Sue's their soul mate;
$F$ says Sam's their soul mate.

**Pop Quiz** How to prove 5(a)?

# Propositions are T or F

We use the letters $p, q, r, s, \ldots$ to represent propositions.

| | |
|---|---|
| $p$: Porky the pig can fly. | F |
| $q$: You got an $A$. | T? |
| $r$: Kilam is an American. | T? |
| $s$: $4^2$ is even. | T |

To get complex statements, combine basic propositions using logical connectors.

# Compound Propositions

$p$: Porky the pig can fly. F

$q$: You got an $A$. T?

$r$: Kilam is an American. T?

$s$: $4^2$ is even. T

| Connector | Symbol | An example in words |
|---|---|---|
| NOT | $\neg p$ | IT IS NOT THE CASE THAT (Porky the pig can fly) |
| AND | $p \wedge q$ | (Porky the pig can fly) AND (You got an $A$) |
| OR | $p \vee q$ | (Porky the pig can fly) OR (You got an $A$) |
| IF... THEN... | $p \rightarrow q$ | IF (Porky the pig can fly) THEN (You got an $A$) |

# Negation (NOT), $\neg p$

The negation $\neg p$ is T when $p$ is F, and the negation $\neg p$ is F when $p$ is T.

"Porky the pig can fly" is F

So,

IT IS NOT THE CASE THAT (Porky the pig can fly) is T

# Conjunction (AND), $p \wedge q$

Both $p$ and $q$ must be T for $p \wedge q$ to be T; otherwise $p \wedge q$ is F.

"Porky the pig can fly" is F

We don't know whether "You got an A".

It does not matter.

(Porky the pig can fly) $\wedge$ (You got an A) is F

# Disjunction (OR), $p \vee q$

**Both $p$ and $q$ must be F for $p \vee q$ to be F; otherwise $p \vee q$ is T.**

"Porky the pig can fly" is F

We don't know whether "You got an A".

Now it matters.

(Porky the pig can fly) $\vee$ (You got an A) is T or F

(Depends on whether you got an A.)

**Pop Quiz:** "You can have cake" OR "You can have ice-cream." Can you have both?

# Truth Tables

| $p$ | $q$ | $\neg p$ | $p \wedge q$ | $p \vee q$ |
|---|---|---|---|---|
| F | F | T | F | F |
| F | T | T | F | T |
| T | F | F | F | T |
| T | T | F | T | T |

The truth table defines the “meaning” of these logical connectors.

# Implication (IF... THEN...), $p \to q$

<u>IF</u> "Porky the pig can fly" <u>THEN</u> "You got an $A$." (T/F?)

Suppose $T$. Since pigs can't fly, does it mean you can't get an A?

<u>IF</u> "$n^2$ is even", <u>THEN</u> "$n$ is even." (T)

Suppose $n^2$ is even. Can we conclude $n \neq 5$?

<u>IF</u> "it rained last night" <u>THEN</u> "the grass is wet." (T)

$$\begin{aligned} p &: \text{ it rained last night} \\ q &: \text{ the grass is wet} \end{aligned}$$

$$p \to q$$

What does it *mean* for this common-sense implication to be true?
What can you conclude? Did it rain last night? Is the grass wet?

# Adding New Information to a True Implication: $p$ is T

IF "it rained last night" THEN "the grass is wet."

$$p \;:\; \text{it rained last night}$$
$$q \;:\; \text{the grass is wet}$$

$$p \to q$$

**Weather report in morning paper: rain last night.** ← new information

| | | | |
|---|---|---|---|
| IF (it rained last night) THEN (the grass is wet) | T | $p \to q$ | T |
| It rained last night (from the weather report) | T | $p$ | T |
| Is the grass wet? | **YES!** | $\therefore\ q$ | T |

For a **true** implication $p \to q$, when $p$ is T, you can conclude $q$ is T.

# Adding New Information to a True Implication: $q$ is T

IF "it rained last night" THEN "the grass is wet."

$$p : \text{it rained last night}$$
$$q : \text{the grass is wet}$$

$$p \rightarrow q$$

**While picking up the morning paper, you see wet grass.** ← new information

| | | | |
|---|---|---|---|
| IF (it rained last night) THEN (the grass is wet) | T | $p \rightarrow q$ | T |
| The grass is wet (from walking outside) | T | $q$ | T |
| Did it rain last night? | ☹ | $\therefore\ p$ | T or F |

For a **true** implication $p \rightarrow q$, when $q$ is T, you **cannot** conclude $p$ is T.

# Adding New Information to a True Implication: $p$ is F

IF "it rained last night" THEN "the grass is wet."

$$p : \text{it rained last night}$$
$$q : \text{the grass is wet}$$

$$p \rightarrow q$$

**Weather report in morning paper: no rain last night.** ← new information

| | | | | |
|---|---|---|---|---|
| IF (it rained last night) THEN (the grass is wet) | T | | $p \rightarrow q$ | T |
| It rained last night (from the weather report) | F | | $p$ | F |
| Is the grass wet? | ☹ | $\therefore$ | $q$ | T or F |

For a **true** implication $p \rightarrow q$, when $p$ is F, you **cannot** conclude $q$ is F.

# Adding New Information to a True Implication: $q$ is F

IF "it rained last night" THEN "the grass is wet."

$$\begin{aligned} p &: \text{ it rained last night} \\ q &: \text{ the grass is wet} \end{aligned}$$

$$p \rightarrow q$$

**While picking up the paper, you see dry grass.** ← new information

| | | | | |
|---|---|---|---|---|
| IF (it rained last night) THEN (the grass is wet) | T | | $p \rightarrow q$ | T |
| It grass is wet (from walking outside) | F | | $q$ | F |
| Did it rain last night? | | $\therefore$ | $p$ | F |

For a **true** implication $p \rightarrow q$, when $q$ is F, you can conclude $p$ is F.

# Implication: Inferences When New Information Comes

For a **true** implication $p \to q$:

When $p$ is T, you can conclude that $q$ is T.

When $q$ is T, you **cannot** conclude $p$ is T.

When $p$ is F, you **cannot** conclude $q$ is F.

When $q$ is F, you can conclude $p$ is F.

$$\text{IF } \underbrace{\text{(Porky the pig can fly)}}_{\text{F}} \text{ THEN } \underbrace{\text{(You got an } A\text{)}}_{\text{can be T or F (phew)}}$$

# Falsifying “IF (it rained last night) THEN (the grass is wet)”

- You are a scientist collecting data to *verify* that the implication is valid (true).
- **One night it rained. That morning the grass was dry.** ← new information
- What do you think about the implication now?

This is a *falsifying scenario*.

IF (it rains) THEN (the grass is wet) ← not T

$p \to q$ is F *only* when $p$ is T and $q$ is F. In all other cases $p \to q$ is T.

# Implication is *Extremely* Important, $p \to q$

All these are $p \to q$ ($p =$ "it rained last night" and $q =$ "the grass is wet"):

| | |
|---|---|
| If it rained last night then the grass is wet. | IF $p$ THEN $q$ |
| It rained last night implies the grass is wet. | $p$ IMPLIES $q$ |
| It rained last night only if the grass is wet. | $p$ ONLY IF $q$ |
| The grass is wet if it rained last night. | $q$ IF $p$ |
| The grass is wet whenever it rains. | $q$ WHENEVER $p$ |

**Truth Tables:**

| $p$ | $q$ | $\neg p$ | $p \wedge q$ | $p \vee q$ | $\boldsymbol{p \to q}$ |
|---|---|---|---|---|---|
| F | F | T | F | F | T |
| F | T | T | F | T | T |
| T | F | F | F | T | F |
| T | T | F | T | T | T |

# Example: IF (you are hungry OR you are thirsty) THEN you visit the cafeteria

$(p \vee q) \rightarrow r$ where

$p$ : you are hungry
$q$ : you are thirsty
$r$ : you visit the cafeteria

- *You are thirsty:* $q$ is T. In both cases $r$ is T.
  (you visit the cafeteria)
- *You did visit the cafeteria:* $r$ is T.
  Are you hungry? We don't know.
  Are you thirsty? We don't know.
  (You accompanied your hungry friend (row 2).)
- *You did not visit the cafeteria:* $r$ is F.
  $p$ and $q$ are both F.
  (You are neither hungry nor thirsty.)

| | $p$ | $q$ | $r$ | $\boldsymbol{(p \vee q) \rightarrow r}$ |
|---|---|---|---|---|
| 1. | F | F | F | T |
| 2. | F | F | T | T |
| 3. | F | T | F | F |
| 4. | F | T | T | T |
| 5. | T | F | F | F |
| 6. | T | F | T | T |
| 7. | T | T | F | F |
| 8. | T | T | T | T |

# Equivalent Compound Statements

| $p$ | $q$ | $p \to q$ | $\neg q \to \neg p$ | $\neg p \vee q$ | $q \to p$ |
|---|---|---|---|---|---|
| F | F | T | T | T | T |
| F | T | T | T | T | F |
| T | F | F | F | F | T |
| T | T | T | T | T | T |
| | | rains $\to$ wet grass | dry grass $\to$ no rain | no rain $\vee$ wet grass | wet grass $\to$ rain |

$$p \to q \overset{\text{eqv}}{\equiv} \neg q \to \neg p \overset{\text{eqv}}{\equiv} \neg p \vee q$$

Order is very important: $p \to q$ and $q \to p$ **do not** mean the same thing.

IF I'm dead, THEN my eyes are closed **vs.** IF my eyes are closed, THEN I'm dead

**Pop Quiz 3.5.** Compound propositions are used for program control flow, especially IF... THEN....

```
if(x > 0 || (y > 1 && x < y))
  Execute some instructions.
```

```
if(x > 0 || y > 1)
  Execute some instructions.
```

Use truth-tables to show that both do the same thing. Which do you prefer and why?

# Proving an Implication: Reasoning Without Facts

$$\text{IF } (n^2 \text{ is even}) \text{ THEN } (n \text{ is even}).$$

$p: \; n^2$ is even
$q: \; n$ is even
$p \to q$

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | $\boxed{\text{F}}$ |
| T | T | T |

What is $n$? How to prove?

We must show that the highlighted row *cannot* occur.

In this row, $q$ is F: $n = 2k + 1$.

$$n^2 = (2k+1)^2 = 2(2k^2 + 2k) + 1$$

$p$ *cannot* be T. This row cannot happen: $p \to q$ is always T. ■

# Quantifiers

EVERY person has A soulmate.

Kilam has some gray hair.
Everyone has some gray hair.
Any map can be colored with 4 colors with adjacent countries having different colors.
Every even integer $n > 2$ is the sum of 2 primes *(Goldbach, 1742)*.
Someone broke this faucet.
There exists a creature with blue eyes and blonde hair.
All cars have four wheels.

These statements are more complex because of *quantifiers*:

EVERY; A; SOME; ANY; ALL; THERE EXISTS.

Compare:

My Ford Escort has four wheels;
ALL cars have four wheels.

# Predicates Are Like Functions

ALL cars have four wheels

Define *predicate* $P(c)$ and its *domain*

$$\begin{aligned} C &= \{c | c \text{ is a car}\} && \leftarrow \text{ set of cars} \\ P(c) &= \text{“car } c \text{ has four wheels”} \end{aligned}$$

“for all $c$ in $C$, the statement $P(c)$ is true.”

$$\forall c \in C : P(c).$$

($\forall$ means “for all”)

| | Predicate | Function |
|---|---|---|
| | $P(c)$ = “car $c$ has four wheels” | $f(x) = x^2$ |
| Input | parameter $c \in C$ | parameter $x \in \mathbb{R}$ |
| Output | **statement** $P(c)$ | **value** $f(x)$ |
| Example | $P$(Jen’s VW) = “car ‘Jen’s VW’ has four wheels” | $f(5) = 25$ |
| | $\forall c \in C : P(c)$ | $\forall x \in \mathbb{R},\ f(x) \geq 0$ |
| Meaning | For all $c \in C$, the statement $P(c)$ is T. | For all $x \in \mathbb{R}$, $f(x)$ is $\geq 0$. |

# There EXISTS a Creature with Blue eyes and Blonde Hair

Define *predicate* $Q(a)$ and its *domain*

$$\begin{aligned} A &= \{a|a \text{ is a creature}\} && \leftarrow \text{ set of creatures} \\ Q(a) &= \text{“}a \text{ has blue eyes and blonde hair”} \end{aligned}$$

“there exists $a$ in $A$ for which the statement $Q(a)$ is true.”

$$\exists a \in A : Q(a).$$

($\exists$ means “there exists”)

$$\begin{aligned} G(a) &= \text{“}a \text{ has blue eyes”} \\ H(a) &= \text{“}a \text{ has blonde hair”} \end{aligned}$$

$$\exists a \in A : \underbrace{(G(a) \wedge H(a))}_{\text{compound predicate}}$$

(When the domain is understood, we don't need to keep repeating it. We write $\exists a : Q(a)$, or $\exists a : (G(a) \wedge H(a))$.)

# Negating Quantifiers

IT IS NOT THE CASE THAT(There is creature with blue eyes and blonde hair)

Same as: "All creatures don't have blue eyes and blonde hair"

$$\neg\Big(\exists a \in A : Q(a)\Big) \overset{\text{eqv}}{\equiv} \forall a \in A : \neg Q(a)$$

IT IS NOT THE CASE THAT(All cars have four wheels)

Same as: "There is a car which does not have four wheels"

$$\neg\Big(\forall c \in C : P(c)\Big) \overset{\text{eqv}}{\equiv} \exists c \in C : \neg P(c)$$

When you take the negation inside the quantifier and negate the predicate, you must switch quantifiers: $\forall \to \exists$, $\exists \to \forall$

# There is a Soulmate for Every Person

Define domains and a predicate.

$$A = \{a \mid a \text{ is an person}\}.$$

$$P(a,b) = \text{“Person } a \text{ has as a soul mate person } b\text{.”}$$

- There is some special person $b$ who is a soul mate to every person $a$.

$$\exists b : \big(\forall a : P(a,b)\big).$$

- For every person $a$, they have there own personal soul mate $b$.

$$\forall a : \big(\exists b : P(a,b)\big).$$

When quantifiers are mixed, the order in which they appear is important for the meaning. Order generally cannot be switched.

# Proofs with Quantifiers

**Claim 1.** $\forall n > 2$ : IF $n$ is even, THEN $n$ is a sum of two primes. *(Goldbach, 1742)*

**Claim 2.** $\exists (a, b, c) \in \mathbb{N}^3 : a^2 + b^2 = c^2$. ($(a, b, c) \in \mathbb{N}^3$ means triples of natural numbers)

**Claim 3.** $\neg\exists (a, b, c) \in \mathbb{N}^3 : a^3 + b^3 = c^3$.

**Claim 4.** $\forall (a, b, c) \in \mathbb{N}^3 : a^3 + b^3 \neq c^3$.

Think about what it would take to prove these claims.