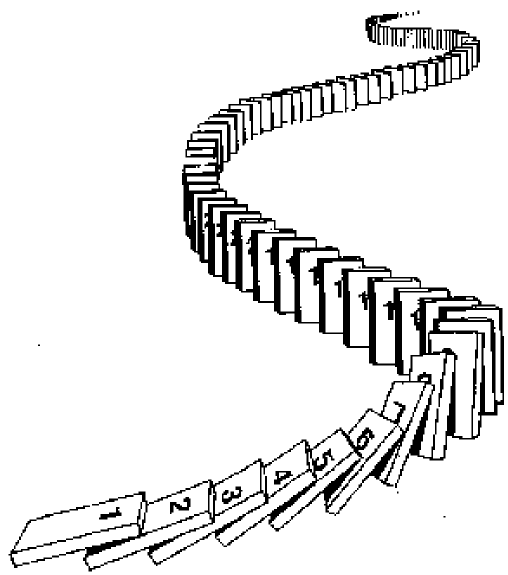
Foundations of Computer Science
Lecture 5

Induction: Proving “For All . . . ”

Induction: What and Why?
Induction: Good, Bad and Ugly
Induction, Well-Ordering and the Smallest Counter-Example



Last Time

1. Proving "IF ..., THEN ...".

2. Proving "...IF AND ONLY IF ...".

3. Proof patterns:
   - direct proof;
     - ★ If $x, y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$.
     - ★ If $4^x - 1$ is divisible by 3, then $4^{x+1} - 1$ is divisible by 3.
   - contraposition;
     - ★ If r is irrational, then $\sqrt{r}$ is irrational.
     - ★ If x^2 is even, then x is even.
   - contradiction.
     - ★ $\sqrt{2}$ is irrational.
     - ★ $a^2 - 4b \neq 2$.
     - ★ $2\sqrt{n} + 1/\sqrt{n+1} \leq 2\sqrt{n+1}$.

Today: Induction, Proving "…for all …"

1. What is induction.
2. Why do we need it?
3. The principle of induction. Toppling the dominos. The induction template.
4. Examples.
5. Induction, Well-Ordering and the Smallest Counter-Example.

Dispensing Postage Using 5¢ and 7¢ Stamps

19¢	20¢	21¢	22¢	23¢
7,7,5	5,5,5,5	7,7,7	5,5,5,7	**?**

Perseverance is a virtue when tinkering.

19¢	20¢	21¢	22¢	23¢	24¢	25¢	26¢	27¢	28¢
7,7,5	5,5,5,5	7,7,7	5,5,5,7	–	7,7,5,5	5,5,5,5,5	7,7,7,5	5,5,5,5,7	7,7,7,7

Can every postage greater than 23¢ can be dispensed?

Intuitively yes.

Induction formalizes that intuition.

Why Do We Need Induction?

	Predicate	Claim
(i)	$P(n) =$ "5¢ and 7¢ stamps can make postage n."	$\forall n \geq 24 : P(n)$
(ii)	$P(n) =$ "$n^2 - n + 41$ a prime number."	$\forall n \geq 1 : P(n)$
(iii)	$P(n) =$ "$4^n - 1$ is divisible by 3."	$\forall n \geq 1 : P(n)$

TINKER!

n	1	2	3	4	5	6	7	8	$\cdots$	40	41
$n^2 - n + 41$	41✓	43✓	47✓	53✓	61✓	71✓	83✓	97✓	$\cdots$	1601✓	1681✘
$(4^n - 1)/3$	1	5	21	85	341	1365	5461	21845	$\cdots$		

How can we prove something for *all* $n \geq 1$? Verification takes too long!

Prove for general n. Can be tricky.

Induction. Systematic.

Is $4^n - 1$ Divisible by 3 for $n \geq 1$?

$$P(n) = \text{“}4^n - 1 \text{ is divisible by 3.”}$$

We proved:

$$\text{IF } \underbrace{4^n - 1 \text{ is divisible by } 3}_{P(n)}, \text{ THEN } \underbrace{4^{n+1} - 1 \text{ is divisible by } 3}_{P(n+1)}.$$

Proof. We prove the claim using a direct proof.

1: Assume that $P(n)$ is T, that is $4^n - 1$ is divisible by 3.
2: This means that $4^n - 1 = 3k$ for an integer k, or that $4^n = 3k + 1$.
3: Observe that $4^{n+1} = 4 \cdot 4^n$, and since $4^n = 3k + 1$, it follows that

$$4^{n+1} = 4 \cdot (3k + 1) = 12k + 4.$$

Therefore $4^{n+1} - 1 = 12k + 3 = 3(4k + 1)$ is a multiple of 3 ($4k + 1$ is an integer).
4: Since $4^{n+1} - 1$ is a multiple of 3, we have shown that $4^{n+1} - 1$ is divisible by 3.
5: Therefore, $P(n + 1)$ is T. ■

We proved:

$$P(n) \rightarrow P(n + 1)$$

What use is this?

(Reasoning in the absense of facts.)

$4^n - 1$ is Divisible by 3 for $n \geq 1$

$$P(n) = \text{“}4^n - 1 \text{ is divisible by 3.”}$$

$$P(n) \to P(n+1)$$

NEW INFORMATION:

From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3$ ← divisible by 3 (new fact)

$$\overset{\checkmark}{P(1)} \to \overset{\checkmark}{P(2)} \to \overset{\checkmark}{P(3)} \to \overset{\checkmark}{P(4)} \to \cdots \to \overset{\checkmark}{P(n-1)} \to \overset{\checkmark}{P(n)} \to \cdots$$

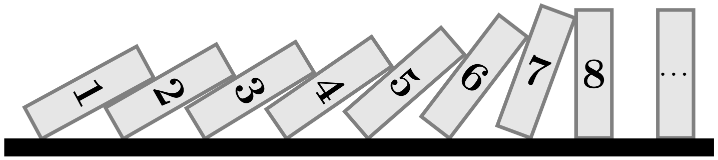
By Induction, $4^n - 1$ is Divisible by 3 for $n \geq 1$

$$P(n) = \text{“}4^n - 1 \text{ is divisible by 3.”}$$

1. $P(1)$ is T.✓
2. $P(n) \rightarrow P(n+1)$ is T.✓

} By induction, $P(n)$ is T for all $n \geq 1$.

$$\boxed{P(1)} \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots$$



$P(n)$ form an infinite chain of dominos.

Topple the first and they *all* fall.

Practice. Exercise 5.2.

Induction Template

Induction to prove: $\forall n \geq 1 : P(n)$.

Proof. We use induction to prove $\forall n \geq 1 : P(n)$.

1: Show that $P(1)$ is T. ("simple" verification.) **[base case]**

2: Show $P(n) \rightarrow P(n+1)$ for $n \geq 1$ **[induction step]**

Prove *the implication* using direct proof or contraposition.

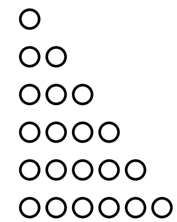
Direct	Contraposition
Assume $P(n)$ is T.	Assume $P(n+1)$ is F.
↓ (valid derivations) must show for any $n \geq 1$ must use $P(n)$ here	↓ (valid derivations) must show for any $n \geq 1$ must use $\neg P(n+1)$ here
Show $P(n+1)$ **is** T.	**Show** $P(n)$ **is** F.

3: Conclude: by induction, $\forall n \geq 1 : P(n)$. ■

- Prove the *implication* $P(n) \rightarrow P(n+1)$ for a *general* $n \geq 1$. (Often direct proof)
  Why is this easier than just proving $P(n)$ for general n?
- Assume $P(n)$ is T, and reformulate it mathematically.
- Somewhere in the proof you *must* use $P(n)$ to prove $P(n+1)$.
- End with a statement that $P(n+1)$ is T.

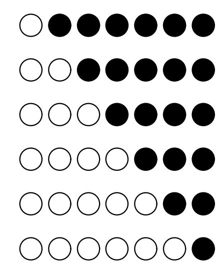
Sum of Integers

$$1 + 2 + 3 + \cdots + (n-1) + n \;=\; \mathbf{?}$$



The GREAT Gauss (age 8-10):

$$\begin{array}{rcccccccc}
S(n) & = & 1 & + & 2 & + & \cdots & + & n \\
S(n) & = & n & + & n-1 & + & \cdots & + & 1 \\
\hline
2S(n) & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) \\
 & = & n \times (n+1) & & & & & &
\end{array}$$



$$S(n) = 1 + 2 + 3 + \cdots + (n-1) + n = \tfrac{1}{2}n(n+1)$$

Proof: $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$

Proof. (By Induction) $P(n) : \sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.

1: **[Base case]** $P(1)$ claims that $1 = \frac{1}{2} \times 1 \times (1+1)$, which is clearly T.

2: **[Induction step]** We show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$, using a direct proof.
Assume (induction hypothesis) $P(n)$ *is* T*:* $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$.
Show $P(n+1)$ *is* T*:* $\sum_{i=1}^{\boldsymbol{n+1}} i = \frac{1}{2}(\boldsymbol{n+1})(\boldsymbol{n+1}+1)$.

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{i} + (n+1) && \text{[key step]} \\
&= \boldsymbol{\tfrac{1}{2}n(n+1)} + (n+1) && \text{[induction hypothesis } P(n)\text{]} \\
&= \tfrac{1}{2}(n+1)(n+2) && \text{[algebra]} \\
&= \tfrac{1}{2}(\boldsymbol{n+1})(\boldsymbol{n+1}+1).
\end{aligned}$$

This is exactly what was to be shown. So, $P(n+1)$ is T.

3: By induction, $P(n)$ is T for all $n \geq 1$. ■

VERY BAD! Induction Step

$\sum^{n+1} i = \frac{1}{2}(n+1)(n+$ … Compare: $\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$

$\sum_{i=1}^{n+1} i -$ … $(n$ … $(n+1)$

$\sum_{i=1}$ … $+2) - (n+1)$

… $)$

… $= \frac{1}{2}n(n+1)$✓ (phew, nothing bad 😊)

	7	=	4	
→	4	=	7	($a{=}b \to b{=}a$)
+	11	=	11✓	(phew 😊)

(Have we proved 7=4?)

To start, you can **NEVER** assert (as though its true) what you are trying to prove.

Sum of Integer Squares

$$S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 \;=\; \textbf{?}$$

Replace Gauss with TINKERING: *method of differences.*

	n	1	2	3	4	5	6	7
	$S(n)$	1	5	14	30	55	91	140
1st difference	$S'(n)$		4	9	16	25	36	49
2nd difference	$S''(n)$			5	7	9	11	13
3rd difference	$S'''(n)$				2	2	2	2

3'rd difference constant is like 3'rd derivative constant. So guess:

$$S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3.$$

$$\begin{array}{rcl} a_0 + a_1 + a_2 + a_3 &=& 1 \\ a_0 + 2a_1 + 4a_2 + 8a_3 &=& 5 \\ a_0 + 3a_1 + 9a_2 + 27a_3 &=& 14 \\ a_0 + 4a_1 + 16a_2 + 64a_3 &=& 30 \\ \hline \end{array}$$

$$a_0 = 0,\ a_1 = \tfrac{1}{6},\ a_2 = \tfrac{1}{2},\ a_3 = \tfrac{1}{3}$$

n	1	2	3	4	5	6	7	8	9	10
$\frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$	1	5	14	30	55	91	140	204	285	385

Proof. (By induction.) $P(n) : \sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

1: **[Base case]** $P(1)$, claims that $1 = \frac{1}{6} \times 1 \times 2 \times 3$, which is clearly T.

2: **[Induction step]** Show $P(n) \to P(n+1)$ for all $n \geq 1$. Direct proof.
Assume (induction hypothesis) $P(n)$ *is* T*:* $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.
Show $P(n+1)$ *is* T*:* $\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$.

$$\begin{aligned} \sum_{i=1}^{n+1} i^2 &= \boldsymbol{\sum_{i=1}^{n} i^2} + (n+1)^2 && \text{[key step]} \\ &= \boldsymbol{\tfrac{1}{6}n(n+1)(2n+1)} + (n+1)^2 && \text{[induction hypothesis } P(n)\text{]} \\ &= \tfrac{1}{6}(n+1)(n+2)(2n+3) && \text{[algebra]} \end{aligned}$$

This is exactly what was to be shown. So, $P(n+1)$ is T.

3: By induction, $P(n)$ is T for all $n \geq 1$. ■

Induction Gone Wrong

$$\boxed{\text{P(1)}} \to P(2) \to P(3) \to P(4) \to P(5) \to P(6) \to P(7) \to \cdots$$

No Base Case.

$$P(1) \to P(2) \to P(3) \to P(4) \to \cdots$$

False: $P(n) : n \le n + 1$ for all $n \ge 1$.

$$n \le n+1 \to n+1 \le n+2 \qquad \text{therefore} \qquad P(n) \to P(n+1).$$

[Every link is proved, but without the base case, you have *nothing*.]

Broken Chain.

$$\boxed{P(1)} \quad P(2) \to P(3) \to P(4) \to \cdots$$

False: $P(n)$: "all balls in any set of n balls are the same color."

Induction step. Suppose any set of n balls have the same color. Consider any set of $n+1$ balls $b_1, b_2, \ldots, b_n, b_{n+1}$. So, $b_1, b_2, \ldots, b_n$ have the same color and $b_2, b_3, \ldots, b_{n+1}$ have the same color. Thus $b_1, b_2, b_3, \ldots, b_{n+1}$ have the same color.

$$P(n) \to P(n+1)?$$

[A *single* broken link kills the entire proof.]

Well Ordering Principle

Well-ordering Principle.
Any non-empty set of natural numbers has a minimum element.

Induction follows from well ordering. Let $P(1)$ and $P(n) \rightarrow P(n+1)$ be $\textsc{t}$.

Suppose $P(n_*)$ fails for the **smallest** counter-example n_* (well-ordering).

$$\boxed{P(1)} \rightarrow \boxed{P(2)} \rightarrow \boxed{P(3)} \rightarrow \boxed{P(4)} \rightarrow \cdots \rightarrow \boxed{P(n_* - 1)} \rightarrow P(n_*) \rightarrow \cdots$$

Now how can $P(n_* - 1) \rightarrow P(n_*)$ be $\textsc{t}$?

Any induction proof can also be done using well-ordering.

Example Well-Ordering Proof: $n < 2^n$ for $n \geq 1$

Proof. [Induction] $P(n) : n < 2^n$.
Base case. $P(1)$ is T because $1 < 2^1$.
Induction. Assume $P(n)$ is T: $n < 2^n$. and show $P(n+1)$ is T: $n+1 < 2^{n+1}$.

$$n + 1 \leq n + n = 2n \leq 2 \times 2^n = 2^{n+1}.$$

Therefore $P(n+1)$ is T and, by induction, $P(n)$ is T for $n \geq 1$. ■

Proof. [Well-ordering] Proof by **contradiction**.
Assume that there is an $n \geq 1$ for which $n \geq 2^n$.
Let n_* be the **minimum** such **counter-example**, $n_* \geq 2^{n_*}$. ← well ordering
Since $1 < 2^1$, $n_* \geq 2$. Since $n_* \geq 2$, $\frac{1}{2}n_* \geq 1$ and so,

$$n_* - 1 \geq n_* - \tfrac{1}{2}n_* = \tfrac{1}{2}n_* \geq \tfrac{1}{2} \times 2^{n_*} = 2^{n_*-1}.$$

So, $n_* - 1$ is a *smaller* counter example. **FISHY!** ■

The **method of minimum counter-example** is very powerful.

TINKER

PRACTICE

Challenge. A circle has $2n$ distinct points, n are red and n are blue. Prove that one can start at a blue point and move clockwise always having passed as many blue points as red.

Practice. All exercises and pop-quizzes in chapter 5.
Strengthen. Problems in chapter 5.