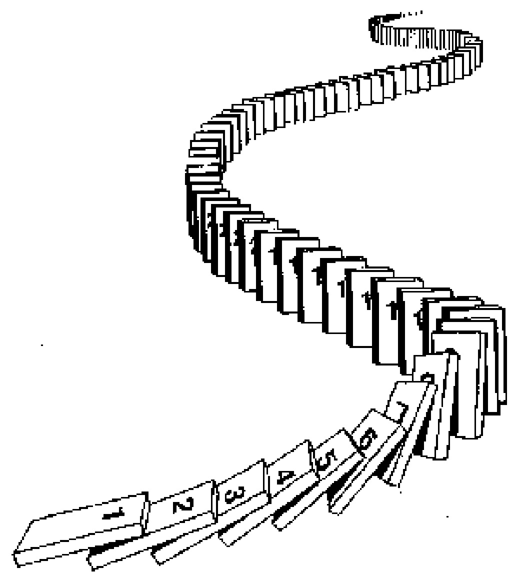
Foundations of Computer Science
Lecture 5

Induction: Proving "For All ..."

Induction: What and Why?

Induction: Good, Bad and Ugly

Induction, Well-Ordering and the Smallest Counter-Example



Last Time

1. Proving "IF ..., THEN ...".

2. Proving "... IF AND ONLY IF ...".

3. Proof patterns:
   - direct proof;
     - ★ If $x, y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$.
     - ★ If $4^x - 1$ is divisible by 3, then $4^{x+1} - 1$ is divisible by 3.
   - contraposition;
     - ★ If r is irrational, then $\sqrt{r}$ is irrational.
     - ★ If x^2 is even, then x is even.
   - contradiction.
     - ★ $\sqrt{2}$ is irrational.
     - ★ $a^2 - 4b \neq 2$.
     - ★ $2\sqrt{n} + 1/\sqrt{n+1} \leq 2\sqrt{n+1}$.

Today: Induction, Proving "...for all ..."

1. What is induction.
2. Why do we need it?
3. The principle of induction. Toppling the dominos. The induction template.
4. Examples.
5. Induction, Well-Ordering and the Smallest Counter-Example.

Dispensing Postage Using 5¢ and 7¢ Stamps

19¢	20¢	21¢	22¢	23¢
7,7,5	5,5,5,5	7,7,7	5,5,5,7	**?**

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Can every postage greater than 23¢ can be dispensed?

Intuitively yes.

Induction formalizes that intuition.

Why Do We Need Induction?

	Predicate	Claim
(i)	$P(n) =$ "5¢ and 7¢ stamps can make postage n."	$\forall n \geq 24 : P(n)$
(ii)	$P(n) =$ "$n^2 - n + 41$ a prime number."	$\forall n \geq 1 : P(n)$
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n	1	2	3	4	5	6	7	8	$\cdots$
$n^2 - n + 41$	41✓	43✓	47✓	53✓	61✓	71✓	83✓	97✓	$\cdots$
$(4^n - 1)/3$	1	5	21	85	341	1365	5461	21845	$\cdots$

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Induction. Systematic.

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Proof. We prove the claim using a direct proof.

1: Assume that $P(n)$ is T, that is $4^n - 1$ is divisible by 3.
2: This means that $4^n - 1 = 3k$ for an integer k, or that $4^n = 3k + 1$.
3: Observe that $4^{n+1} = 4 \cdot 4^n$, and since $4^n = 3k + 1$, it follows that

$$4^{n+1} = 4 \cdot (3k + 1) = 12k + 4.$$

Therefore $4^{n+1} - 1 = 12k + 3 = 3(4k + 1)$ is a multiple of 3 ($4k + 1$ is an integer).
4: Since $4^{n+1} - 1$ is a multiple of 3, we have shown that $4^{n+1} - 1$ is divisible by 3.
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What use is this?

(Reasoning in the absense of facts.)

$4^n - 1$ is Divisible by 3 for $n \geq 1$

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$$P(n) \to P(n+1)$$

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From tinkering we know that $P(1)$ is T: $4^1 - 3 = 3$ $\leftarrow$ divisible by 3 (new fact)

✓ $P(1)$

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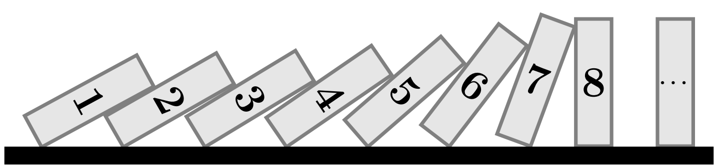
$$\boxed{P(1)} \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow P(5) \rightarrow \cdots$$



$P(n)$ form an infinite chain of dominos.
Topple the first and they *all* fall.

Practice. Exercise 5.2.

Induction Template

Induction to prove: $\forall n \geq 1 : P(n)$.

Proof. We use induction to prove $\forall n \geq 1 : P(n)$.

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Prove *the implication* using direct proof or contraposition.

Direct	Contraposition
Assume $P(n)$ is T.	Assume $P(n+1)$ is F.
↓ (valid derivations)	↓ (valid derivations)
must show for any $n \geq 1$	must show for any $n \geq 1$
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  Why is this easier than just proving $P(n)$ for general n?

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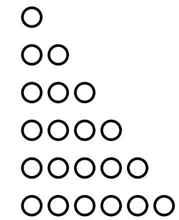
- Assume $P(n)$ is T, and reformulate it mathematically.
- Somewhere in the proof you *must* use $P(n)$ to prove $P(n+1)$.
- End with a statement that $P(n+1)$ is T.

Sum of Integers

$$1 + 2 + 3 + \cdots + (n-1) + n \;=\; \textbf{?}$$



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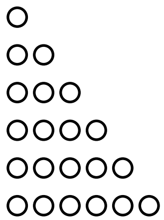
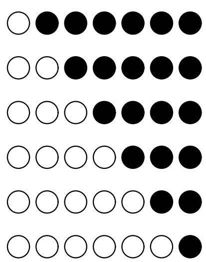


The GREAT Gauss (age 8-10):

$$\begin{array}{rcccccccc} S(n) & = & 1 & + & 2 & + & \cdots & + & n \\ S(n) & = & n & + & n-1 & + & \cdots & + & 1 \\ \hline 2S(n) & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) \\ & = & n \times (n+1) & & & & & & \end{array}$$



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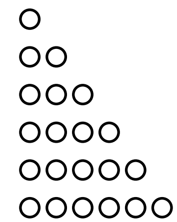

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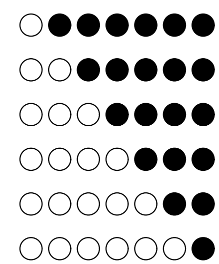

$$S(n) = 1 + 2 + 3 + \cdots + (n-1) + n = \tfrac{1}{2} n(n+1)$$

Proof: $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$

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3: By induction, $P(n)$ is T for all $n \geq 1$. ∎

VERY BAD! Induction Step

$$\sum_{i=1}^{n+1} i \;=\; \tfrac{1}{2}(n+1)(n+2)$$ (What we want)

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(Have we proved 7=4?)

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Compare: $\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$

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To start, you can **NEVER** assert (as though its true) what you are trying to prove.

Sum of Integer Squares

$$S(n) = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 \;\; = \;\; \textbf{?}$$

Where's the GREAT Gauss when you need him?

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Replace Gauss with TINKERING: *method of differences.*

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n	1	2	3	4	5	6	7	8	9	10
$\frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$	1	5	14	30	55	91	140	204	285	385

Proof: $S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n+1)(2n+1)$

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$$= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2 \quad \text{[induction hypothesis } P(n)\text{]}$$

$$= \frac{1}{6}(n+1)(n+2)(2n+3) \quad \text{[algebra]}$$

This is exactly what was to be shown. So, $P(n+1)$ is T.

3: By induction, $P(n)$ is T for all $n \geq 1$. ■

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$$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow P(4) \rightarrow \cdots$$

False: $P(n) : n \leq n+1$ for all $n \geq 1$.

$$n \leq n+1 \rightarrow n+1 \leq n+2 \qquad \text{therefore} \qquad P(n) \rightarrow P(n+1).$$

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$$\boxed{P(1)} \quad P(2) \to P(3) \to P(4) \to \cdots$$

False: $P(n)$: "all balls in any set of n balls are the same color."

> **Induction step.** Suppose any set of n balls have the same color. Consider any set of $n+1$ balls $b_1, b_2, \ldots, b_n, b_{n+1}$. So, $b_1, b_2, \ldots, b_n$ have the same color and $b_2, b_3, \ldots, b_{n+1}$ have the same color. Thus $b_1, b_2, b_3, \ldots, b_{n+1}$ have the same color.
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[A *single* broken link kills the entire proof.]

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Any induction proof can also be done using well-ordering.

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The **method of minimum counter-example** is very powerful.

TINKER

PRACTICE

Challenge. A circle has $2n$ distinct points, n are red and n are blue. Prove that one can start at a blue point and move clockwise always having passed as many blue points as red.

Practice. All exercises and pop-quizzes in chapter 5.
Strengthen. Problems in chapter 5.