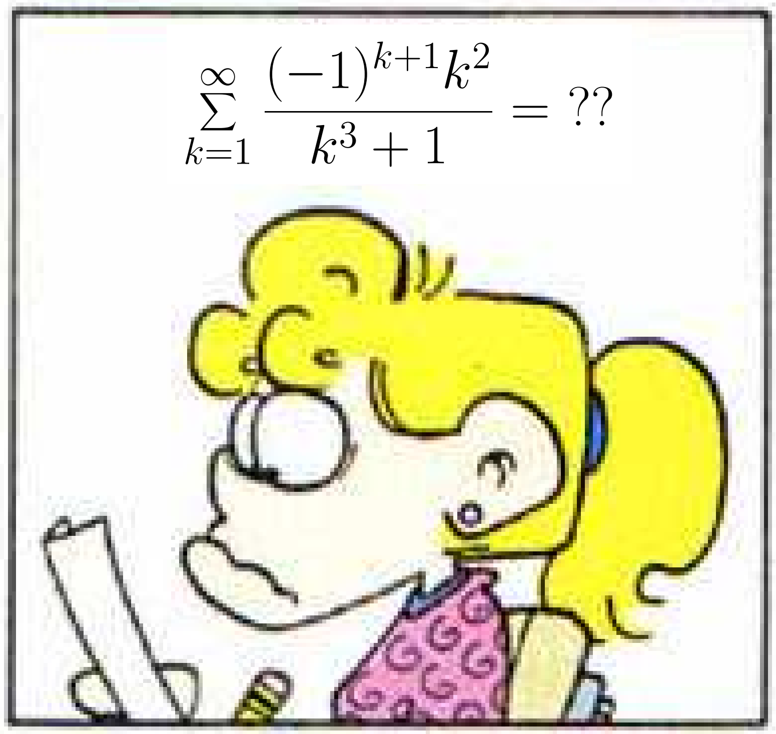
Foundations of Computer Science
Lecture 9

Sums And Asymptotics

Computing Sums
Asymptotics: big-$\Theta(\cdot)$, big-$O(\cdot)$, big-$\Omega(\cdot)$
The Integration Method



Last Time

1. Structural induction: proofs about recursively defined sets.
   - Matched parentheses.
   - $\mathbb{N}$
   - Palindromes.
   - Arithmetic expressions.
   - Rooted Binary Trees (RBT).

Today: Sums And Asymptotics

1. Maximum Substring Sum
2. Computing Sums
3. Asymptotics: Big-Theta, Big-Oh and Big-Omega
4. Integration Method

War Story. A startup asked me about their landing webpage. Users stay 1min, or 2min (about half as many), and so on. To "convert" the 10min user is very different from "converting" the 1 min user. Who should they tailor the webpage to?

Maximum Substring Sum

$$1 \quad -1 \quad -1 \quad \underbrace{2 \quad 3 \quad 4 \quad -1 \quad -1 \quad 2 \quad 3}_{\text{max. substring sum}=\,12} \quad -4 \quad 1 \quad 2 \quad -1 \quad -2 \quad 1$$

More generally, compute the maximum substring sum for

$$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \cdots \quad a_{n-1} \quad a_n$$

(n measures the "size" of the input).

$$T_1(n) = 2 + \sum_{i=1}^{n}\left[2 + \sum_{j=i}^{n}\left(5 + \sum_{k=i}^{j} 2\right)\right]. \qquad \text{(3 \texttt{for} loops)}$$

$$T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right). \qquad \text{(2 \texttt{for} loops)}$$

$$T_3(n) = \begin{cases} 3 & n = 1; \\ 2T_3(\frac{1}{2}n) + 6n + 9 & n > 1 \text{ and even}; \\ T(\frac{1}{2}(n+1)) + T(\frac{1}{2}(n-1)) + 6n + 9 & n > 1 \text{ and odd}. \end{cases} \qquad \text{(recursive)}$$

$$T_4(n) = 5 + \sum_{i=1}^{n} 10. \qquad \text{(1 \texttt{for} loops)}$$

(What does $\sum_{i=1}^{n}$ mean: Pop Quiz 9.1)

Different algorithms have different runtime. Which is best?

Evaluate the Runtimes

$$T_1(n) = 2 + \sum_{i=1}^{n}\Big[2 + \sum_{j=i}^{n}\Big(5 + \sum_{k=i}^{j} 2\Big)\Big].$$

$$T_2(n) = 2 + \sum_{i=1}^{n}\Big(3 + \sum_{j=i}^{n} 6\Big)$$

$$T_3(n) = \begin{cases} 3 & n = 1; \\ 2T_3(\frac{1}{2}n) + 6n + 9 & n > 1 \text{ and even;} \\ T(\frac{1}{2}(n+1)) + T(\frac{1}{2}(n-1)) + 6n + 9 & n > 1 \text{ and odd.} \end{cases}$$

$$T_4(n) = 5 + \sum_{i=1}^{n} 10$$

n	1	2	3	4	5	6	7	8	9	10
$T_1(n)$	11	29	58	100	157	231	324	438	575	737
$T_2(n)$	11	26	47	74	107	146	191	242	299	362
$T_3(n)$	3	27	57	87	123	159	195	231	273	315
$T_4(n)$	15	25	35	45	55	65	75	85	95	105

T_2 is better than T_1; T_2 versus T_3??? What about T_4?

We need:

1. Simple formulas for $T_1(n), \ldots, T_4(n)$: we need to compute sums and solve recurrences.
2. A way to compare runtime-*functions* that captures the essence of the algorithm.

Computing Sums: Tool 1: Constant Rule

$$S_1 = \sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3 \qquad 3 \times 10$$

$$S_2 = \sum_{i=1}^{10} j = j+j+j+j+j+j+j+j+j+j \qquad j \times 10$$

$$S_3 = \sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10 \qquad \tfrac{1}{2} \times 10 \times (10+1)$$

The *index of summation* is i in these examples.

Constants (independent of summation index) can be taken outside the sum.

$$S_1 = \sum_{i=1}^{10} 3 = 3\sum_{i=1}^{10} 1 = 3 \times 10 \qquad S_2 = \sum_{i=1}^{10} j = j\sum_{i=1}^{10} 1 = j \times 10.$$

Pop Quiz 9.2 Compute $T_4(n) = 5 + \sum_{i=1}^{n} 10$.

Computing Sums: Tool 2: Addition Rule

$$
\begin{aligned}
S &= \sum_{i=1}^{5}(i + i^2) \\
&= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) + (4 + 4^2) + (5 + 5^2) \\
&= (1 + 2 + 3 + 4 + 5) + (1^2 + 2^2 + 3^2 + 4^2 + 5^2) && \text{(rearrange terms)} \\
&= \sum_{i=1}^{5} i + \sum_{i=1}^{5} i^2.
\end{aligned}
$$

The sum of terms added together is the addition of the individual sums.

$$\sum_i (a(i) + b(i) + c(i) + \cdots) = \sum_i a(i) + \sum_i b(i) + \sum_i c(i) + \cdots$$

Computing Sums: Tool 3: Common Sums

$$\sum_{i=k}^{n} 1 = n+1-k \qquad \sum_{i=1}^{n} i = \tfrac{1}{2}n(n+1) \qquad \sum_{i=0}^{n} 2^i = 2^{n+1}-1$$

$$\sum_{i=1}^{n} f(x) = nf(x) \qquad \sum_{i=1}^{n} i^2 = \tfrac{1}{6}n(n+1)(2n+1) \qquad \sum_{i=0}^{n} \frac{1}{2^i} = 2-\frac{1}{2^n}$$

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \quad {\scriptstyle (r\neq 1)} \qquad \sum_{i=1}^{n} i^3 = \tfrac{1}{4}n^2(n+1)^2 \qquad \sum_{i=1}^{n} \log i = \log n!$$

Example: $\sum_{i=1}^{n}(1+2i+2^{i+2})$

$$\begin{aligned}
\sum_{i=1}^{n}(1+2i+2^{i+2}) &= \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^{i+2} && \text{(addition rule)}\\
&= \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i + 4\sum_{\mathbf{i=1}}^{n} 2^i && \text{(constant rule)}\\
&= n + 2\times\tfrac{1}{2}n(n+1) + 4\cdot(2^{n+1}-1\mathbf{-1}) && \text{(common sums)}\\
&= n + n(n+1) + 2^{n+3} - 8 && \text{(common sums)}
\end{aligned}$$

Computing Sums: Tool 3: Nested Sum Rule

$$S_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} 1; \qquad S_2 = \sum_{i=1}^{3} \sum_{j=1}^{i} 1.$$

To compute a nested sum, start with the innermost sum and proceed outward.

$$S_1 = \underset{(i=1)}{\sum_{j=1}^{3} 1} + \underset{(i=2)}{\sum_{j=1}^{3} 1} + \underset{(i=3)}{\sum_{j=1}^{3} 1} = 3+3+3 = 9.$$

$$S_2 = \underset{(i=1)}{\sum_{j=1}^{1} 1} + \underset{(i=2)}{\sum_{j=1}^{2} 1} + \underset{(i=3)}{\sum_{j=1}^{3} 1} = 1+2+3 = 6.$$

More generally:

$$S(n) = \sum_{i=1}^{n} \sum_{j=1}^{i} 1 = \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{i} 1}_{f(i)=i} = \sum_{i=1}^{n} i = \tfrac{1}{2} n(n+1).$$

Computing a Formula for $T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right)$

$$
\begin{aligned}
T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right) &= 2 + \sum_{i=1}^{n} 3 + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 && \text{(sum rule)} \\
&= 2 + 3\sum_{i=1}^{n} 1 + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 && \text{(constant rule)} \\
&= 2 + 3n + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 && \text{(common sum)} \\
&= 2 + 3n + \sum_{i=1}^{n}\sum_{j=i}^{n} 6 && \text{(innermost sum)} \\
&= 2 + 3n + 6\sum_{i=1}^{n}\sum_{j=i}^{n} 1 && \text{(constant rule)} \\
&= 2 + 3n + 6\sum_{i=1}^{n}(n + 1 - i) && \text{(common sum)} \\
&= 2 + 3n + 6(n + (n-1) + \cdots + 1) && \\
&= 2 + 3n + 6 \times \tfrac{1}{2}n(n+1) && \text{(common sum)} \\
&= 2 + 6n + 3n^2 && \text{(algebra)}
\end{aligned}
$$

Practice: Compute a Formula for the Sum $\sum_{i=1}^{n}\sum_{j=1}^{i} ij$

$$\sum_{i=1}^{n}\sum_{j=1}^{i} ij = \sum_{i=1}^{n}\sum_{j=1}^{i} ij \qquad \text{(innermost sum)}$$

$$= \sum_{i=1}^{n} i \sum_{j=1}^{i} j \qquad \text{(constant rule)}$$

$$= \sum_{i=1}^{n} i \times \tfrac{1}{2}i(i+1) \qquad \text{(common sum)}$$

$$= \tfrac{1}{2}\sum_{i=1}^{n}(i^3+i^2) \qquad \text{(algebra, constant rule)}$$

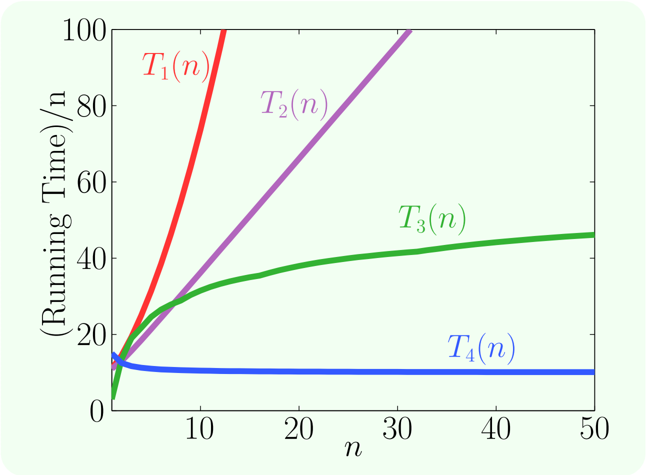
$$= \tfrac{1}{2}\sum_{i=1}^{n} i^3 + \tfrac{1}{2}\sum_{i=1}^{n} i^2 \qquad \text{(sum rule)}$$

$$= \tfrac{1}{8}n^2(n+1)^2 + \tfrac{1}{12}n(n+1)(2n+1) \qquad \text{(common sums)}$$

$$= \tfrac{1}{12}n + \tfrac{3}{8}n^2 + \tfrac{5}{12}n^3 + \tfrac{1}{8}n^4 \qquad \text{(algebra)}$$

Summary of Maximum Substring Sum Algorithms

Runtimes
$\boldsymbol{T_1(n)} = 2 + \frac{31}{6}n + \frac{7}{2}n^2 + \frac{1}{3}n^3$
$\boldsymbol{T_2(n)} = 2 + 6n + 3n^2$
$3n(\log_2 n + 1) - 9 \leq \boldsymbol{T_3(n)} \leq 12n(\log_2 n + 3) - 9$
$\boldsymbol{T_4(n)} = 5 + 10n$
(“simple” formulas for $T_1(n), \ldots, T_4(n)$)



So, which algorithm is best?

Computers solve problems with big inputs. We care about large n.

- Compare runtimes *asymptotically* in the input size n. That is $n \to \infty$.
- Ignore additive and multiplicative constants (minutia). We care about *growth rate*.

Algorithm 4 is *linear* in n, $\frac{T_4(n)}{n} \to$ constant.

Asymptotically Linear Functions: $\Theta(n)$, big-Theta-of-n

$T \in \Theta(\boldsymbol{n})$, if there are positive constants c, C for which
$$c \cdot \boldsymbol{n} \le T(n) \le C \cdot \boldsymbol{n}.$$

$$\frac{T(n)}{n} \xrightarrow[n\to\infty]{} \begin{cases} \infty & T \in \omega(n), & \text{“}T > n\text{”}; \\ \text{constant}>0 & T \in \Theta(n), & \text{“}T = n\text{”}; \\ 0 & T \in o(n), & \text{“}T < n\text{”}. \end{cases}$$

Linear means in $\Theta(n)$:

$$2n+7, \qquad 2n+15\sqrt{n}, \qquad 10^9 n + 3, \qquad 3n + \log n, \qquad 2^{\log_2 n + 4}.$$

Not linear, not in $\Theta(n)$:

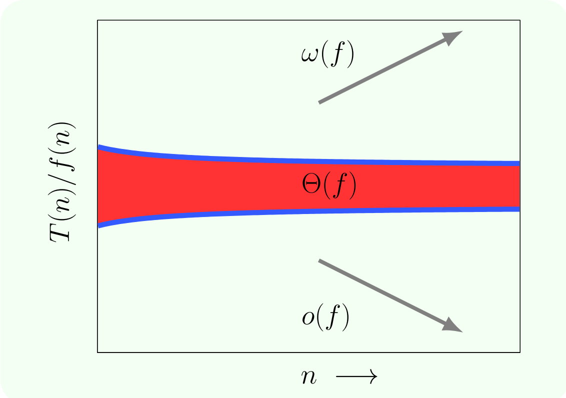
$$10^{-9}n^2, \quad 10^9\sqrt{n}+15, \quad n^{1.0001}, \quad n^{0.9999}, \quad n\log n, \quad \frac{n}{\log n}, \quad 2^n.$$

Other runtimes from practice:

log	linear	loglinear	quadratic	cubic	superpolynomial	exponential	factorial	BAD
$\Theta(\log n)$	$\Theta(n)$	$\Theta(n\log n)$	$\Theta(n^2)$	$\Theta(n^3)$	$\Theta(n^{\log n})$	$\Theta(2^n)$	$\Theta(n!)$	$\Theta(n^n)$

General Asymptotics: $\Theta(f)$, big-Theta-of-f

$$\frac{T(n)}{f(n)} \xrightarrow[n\to\infty]{} \begin{cases} \infty & T \in \omega(f),\ \text{“}T < f\text{”}; \\ \text{constant}>0 & T \in \Theta(f),\ \text{“}T = f\text{”}; \\ 0 & T \in o(f),\ \text{“}T < f\text{”}. \end{cases}$$



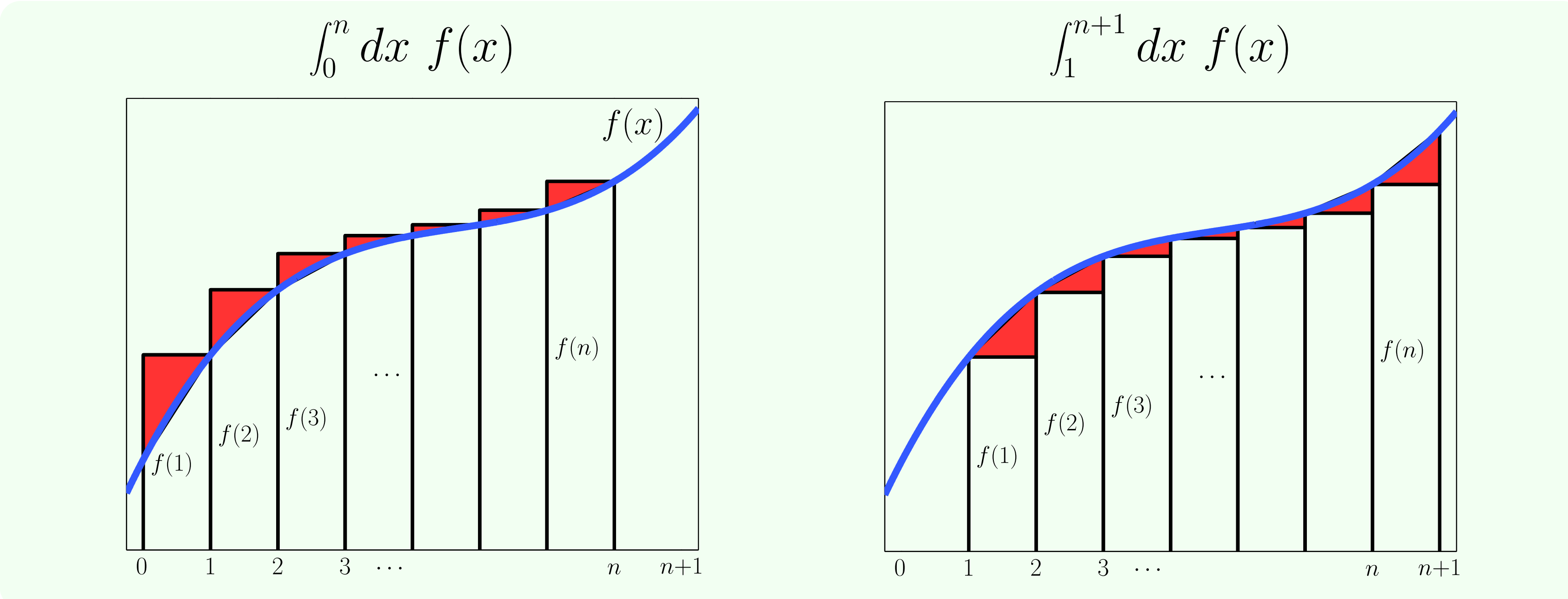
$T \in o(f)$	$T \in O(f)$	$T \in \Theta(f)$	$T \in \Omega(f)$	$T \in \omega(f)$
“$T < f$”	“$T \leq f$”	“$T = f$”	“$T \geq f$”	“$T > f$”
	$T(n) \leq Cf(n)$	$cf(n) \leq T(n) \leq Cf(n)$	$cf(n) \leq T(n)$	

Examples and Practice. (See also Exercise 9.6)

- For polynomials, growth rate is the highest order.
- For nested sums, growth rate is number of nestings plus order of summand.

$2n^2$	$n^2 + n\sqrt{n}$	$n^2 + \log^{256} n$	$n^2 + n^{1.99}\log^{256} n$	$\sum_{i=1}^{n} i$	$\sum_{i=1}^{n}\sum_{j=1}^{i} 1$	$\sum_{i=1}^{n}\sum_{j=1}^{i} ij$
$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^2)$	$\Theta(n^4)$

The Integration Method



Theorem. (Integration Bound)

For a monotonically increasing function f,

$$\int_{m-1}^{n} dx\ f(x) \leq \sum_{i=m}^{n} f(i) \leq \int_{m}^{n+1} dx\ f(x).$$

(If f is monotonically decreasing, the inequalities are reversed.)

Integration For Quickly Getting Asymptotic Behavior

- **Integer Powers.** Set $f(x) = x^k$:

$$\sum_{i=1}^{n} i^k \approx \int_0^n dx\ x^k = \frac{n^{k+1}}{k+1} \in \Theta(n^{k+1}).$$

- **Harmonic Numbers.** Set $f(x) = 1/x$ (monotonically decreasing):

$$\underbrace{\int_1^{n+1} dx\ \tfrac{1}{x}}_{\ln(n+1)} \quad \le \quad H_n = \sum_{i=1}^{n} \frac{1}{i} \quad \le \quad \underbrace{1 + \int_1^{n} dx\ \tfrac{1}{x}}_{1+\ln n}.$$

- **Stirling's Approximation for $\ln n!$.** Set $f(x) = \ln x$:

$$\ln n! = \sum_{i=1}^{n} \ln i \quad \le \quad \int_1^{n+1} dx\ \ln x = (n+1)\ln(n+1) - n \in \Theta(n \ln n).$$

- **Analyzing a Recurrence** $\quad T_1 = 1; \quad T_n = T_{n-1} + n\sqrt{n} - \ln n.$

First unfold the recurrence

$$\begin{aligned}
T_n &= \cancel{T_{n-1}} + n\sqrt{n} - \ln n \\
\cancel{T_{n-1}} &= \cancel{T_{n-2}} + (n-1)\sqrt{n-1} - \ln(n-1) \\
&\ \vdots \\
\cancel{T_3} &= \cancel{T_2} + 3\sqrt{3} - \ln 3 \\
\cancel{T_2} &= \cancel{T_1}^{\,1} + 2\sqrt{2} - \ln 2 \\
\hline
+\ T_n &= 1 + 2\sqrt{2} + \cdots + n\sqrt{n} - (\ln 2 + \ln 3 + \cdots + \ln n) \\
&= \sum_{i=1}^{n} i\sqrt{i} - \sum_{i=1}^{n} \ln i
\end{aligned}$$

$$T(n) = \underbrace{\sum_{i=1}^{n} i\sqrt{i}}_{\Theta(n^{5/2})} - \underbrace{\sum_{i=1}^{n} \ln i}_{\ln n! \in \Theta(n \ln n)}$$

$$T(n) \in \Theta(n^{5/2}).$$