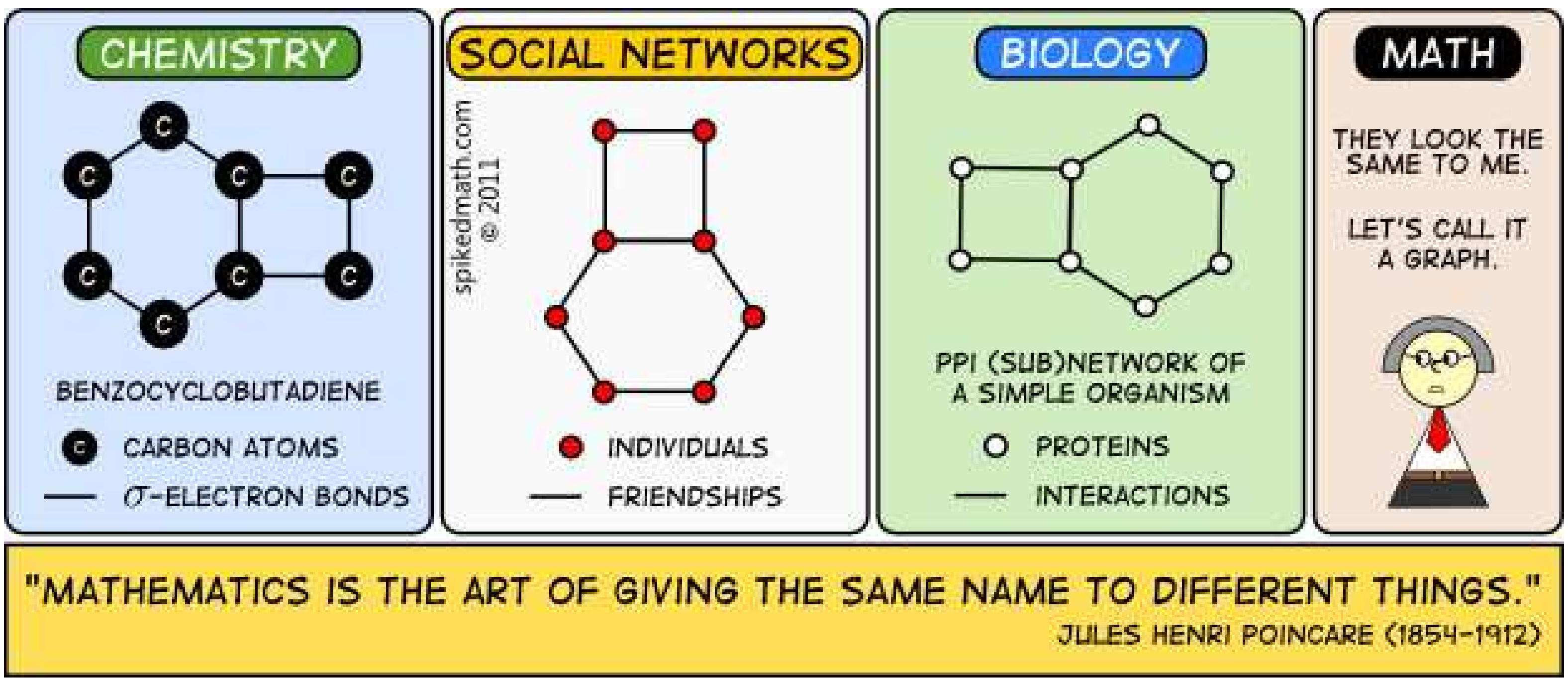
Foundations of Computer Science
Lecture 11

Graphs

Definition and Properties. Equivalence of Graphs.
Degree Sequences and The Handshaking Theorem.
Planar Graphs.
Different Types of Graphs: Multigraph, Weighted, Directed.



Last Time

1. Division, quotient and remainder. Properties of divisibility.
2. Greatest common divisor and Euclid's algorithm.
   - Bezout's Identity: The GCD is the smallest linear combination.
   - Euclid's Lemma: $p|q_1 \cdots q_\ell \rightarrow p$ is one of the q_i.
3. Fundamental Theorem of Arithmetic Part II: Uniqeness of prime factorization.
4. Modular arithmetic
   - **Pop Quiz:** What is the last digit of 29^{29}.
5. RSA

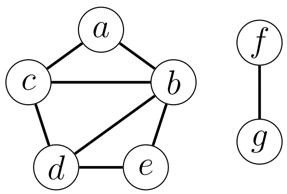
Today: Graphs

1. Graph basics and notation
   - Equivalent graphs: isomorphism
2. Degree sequence
   - Handshaking Theorem
3. Trees
4. Planar graphs
5. Other types of graphs: multigraph, weighted, directed
6. Problem solving with Graphs

Graph Basics and Notation

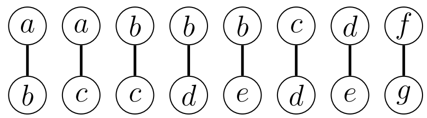
Graphs model relationships: friendships (e.g. social networks)
connectivity (e.g. cities linked by highways)
conflicts (e.g. radio-stations with listener overlap)

Graph G



Vertices (aka nodes): ⓐ ⓑ ⓒ ⓓ ⓔ ⓕ ⓖ $\qquad V = \{a, b, c, d, e, f, g\}$.

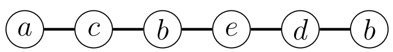
Edges:



$$E = \left\{ \begin{matrix} (a, b), (a, c), (b, c), (b, d), \\ (b, e), (c, d), (d, e), (f, g) \end{matrix} \right\}.$$

Degree: Number of relationships $\qquad$ *e.g.*, $\text{degree}(b) = 4$.

Path:

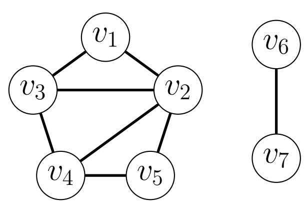


$p = acbedb$.

Graph Isomorphism. Relabeling the nodes in G to $v_1, \ldots, v_7$.

$$\begin{aligned} a &\to v_1, \\ b &\to v_2, \\ c &\to v_3, \\ d &\to v_4, \\ e &\to v_5, \\ f &\to v_6, \\ g &\to v_7 \end{aligned}$$

Relabeling of Graph G



$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}.$$

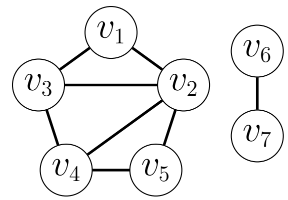
$$E = \left\{ \begin{matrix} (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), \\ (v_2, v_5), (v_3, v_4), (v_4, v_5), (v_6, v_7) \end{matrix} \right\}.$$

If two graphs can be relabeled with $v_1, \ldots, v_n$, giving the *same* edge set, they are equivalent – *isomorphic*.

Practice. Pop Quiz 11.1; Exercise 11.2.

Paths and Connectivity

Graph, G



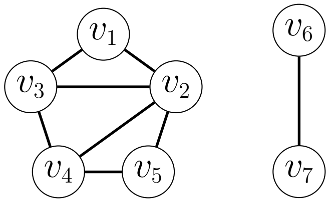
- A *path* from v_1 to v_2 is a sequence of vertices (start is v_1 and end is v_2): $v_1v_3v_2v_5v_4v_2$
- There is an edge in the graph between consecutive vertices in the path.

  v_1 and v_2 are *connected.*

- The length of a path is the number of edges traversed (5).
- *Cycle:* path that starts and ends at a vertex without repeating any edge: $v_1v_2v_3v_1$
- v_1 and v_6 are not connected by a path.
- The graph G is *not* connected (*every* pair of vertices must be connected by a path).
- How can we make G connected?

Graph Representation

Graph



Adjacency List

v_1: v_2, v_3
v_2: v_1, v_3, v_4, v_5
v_3: v_1, v_2, v_4
v_4: v_2, v_3, v_5
v_5: v_2, v_4
v_6: v_7
v_7: v_6

Adjacency Matrix

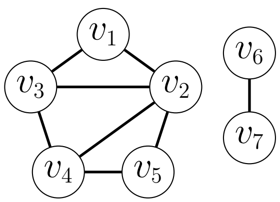
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

More wasted memory; faster algorithms.

Small redundancy: every edge is "represented" twice.

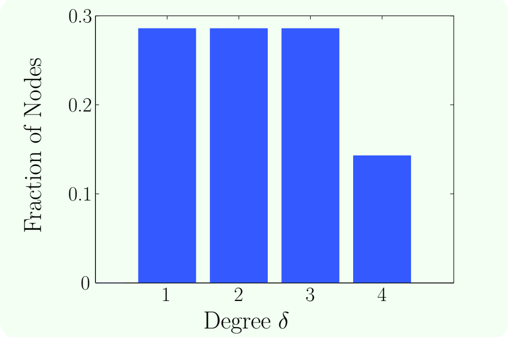
Degree Sequence

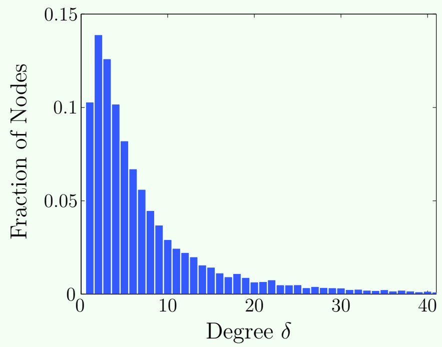
<u>Graph</u>



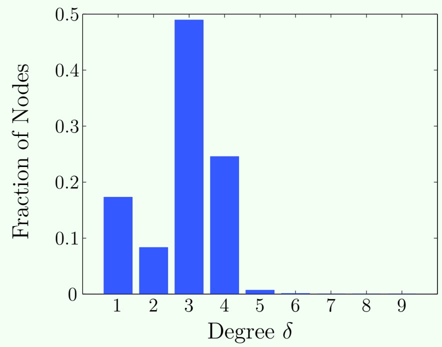
$$\text{degree } \delta_i \;=\; \text{number of } v_i\text{'s neighbors}$$
$$=\; \sum_{j=1}^{n} \mathrm{A}_{ij}.$$

$$\begin{array}{ccccccccc} & & v_2 & v_3 & v_4 & v_1 & v_5 & v_6 & v_7 \\ \boldsymbol{\delta} & = & [\;4 & 3 & 3 & 2 & 2 & 1 & 1\;] \end{array}$$

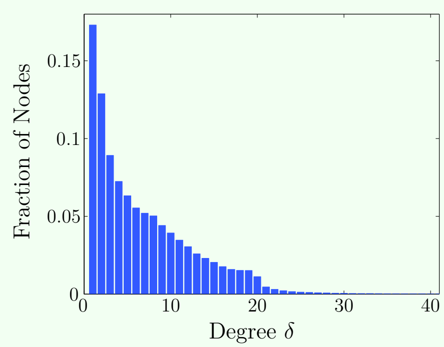




Co-author network

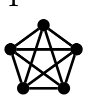


PA road network



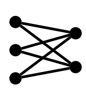
Web graph

Complete, K_5



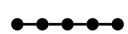
[4, 4, 4, 4, 4]

Bipartite, $K_{3,2}$



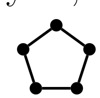
[3, 3, 2, 2, 2]

Line, L_5



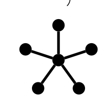
[2, 2, 2, 1, 1]

Cycle, C_5



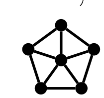
[2, 2, 2, 2, 2]

Star, S_6



[5, 1, 1, 1, 1, 1]

Wheel, W_6



[5, 3, 3, 3, 3, 3]

Handshaking Theorem

Pop Quiz. Construct a graph with degree sequence $\boldsymbol{\delta} = [3, 3, 3, 2, 1, 1]$.

Theorem. Handshaking Theorem

For any graph the sum of vertex-degrees equals twice the number of edges, $\sum_{i=1}^{n} \delta_i = 2|E|$.

Proof. Every edge contributes 2 to the sum of degrees. (Why?)
If there are $|E|$ edges, their contribution to the sum of degrees is $2|E|$. ∎

Exercise. Give a formal proof by induction on the number of edges in the graph.

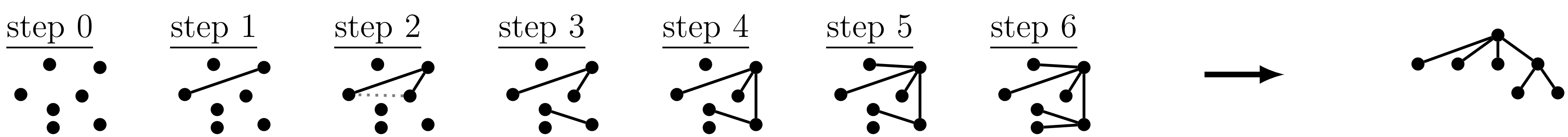
Pop Quiz (Answer). Can't be done: sum of degrees is $3 + 3 + 3 + 2 + 1 + 1 = 13$ (odd).

Exercise. At a party a person is odd if they shake hands with an odd number of people. Show that the number of odd people is even.

Trees (More General than RBTs)

Definition: General Tree.
A tree is a *connected* graph with no cycles.

Building a tree, one edge at a time.

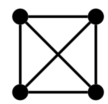


Exercise 11.6. Every tree with n vertices has $n-1$ edges. (We proved this for RBTs.)

Planar Graphs

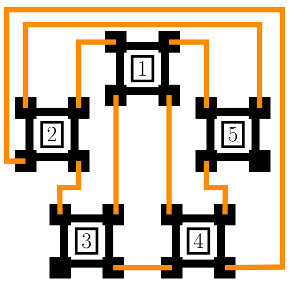
A graph is planar if you can draw it without edge crossings.

Complete graph K_4:

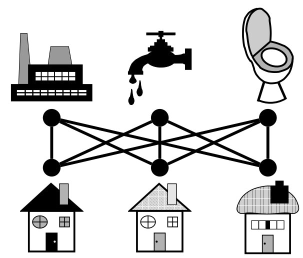


non planar drawing

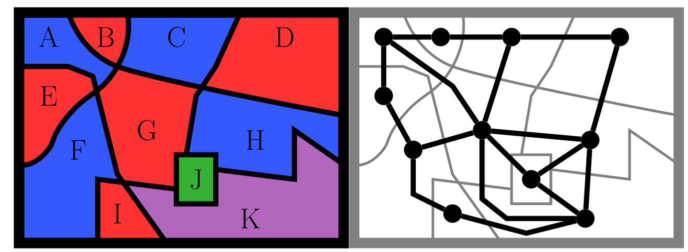
planar drawings → K_4 is planar



Chip design: CPUs must be connected without wire-crossings.



Town planing: connect utilities to homes without pipe-crossings.



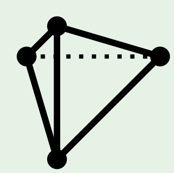
Map coloring: adjacent countries sharing a border must have different colors. The map corresponds to a planar graph.

Exercise 11.7. Euler's Invariant Characteristic: $F + V - E = 2$.

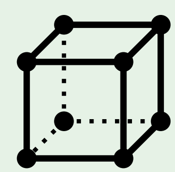
(Faces, F: outer region or region enclosed by a cycle.)

	V	E	F	$F + V - E$
planar K_4	4	6	4	$4 + 4 - 6 = 2$ ✓
planar map	11	17	8	$8 + 11 - 17 = 2$ ✓
pyramid	4	6	4	$4 + 4 - 6 = 2$ ✓
cube	8	12	6	$6 + 8 - 12 = 2$ ✓
octohedron	6	12	8	$8 + 6 - 12 = 2$ ✓

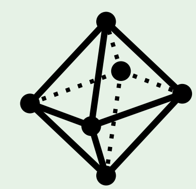
Pyramid



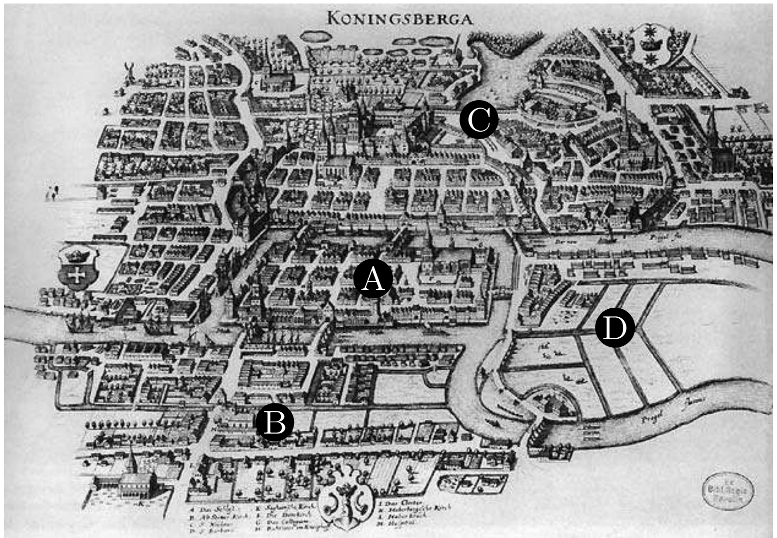
Cube

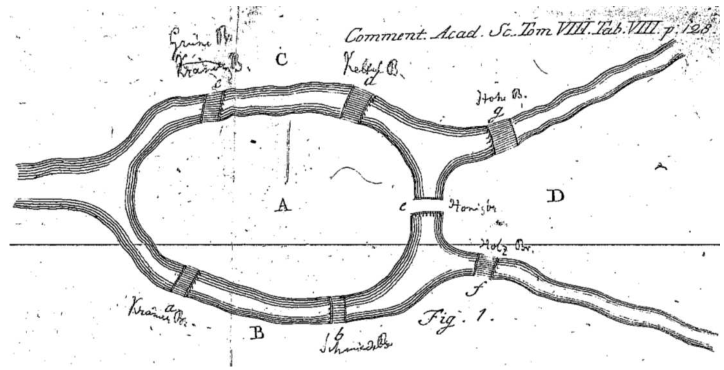


Octohedron

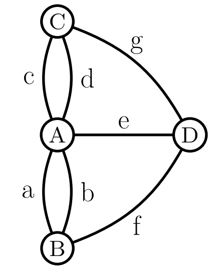


Other Types of Graphs: Multigraph, Weighted, Directed

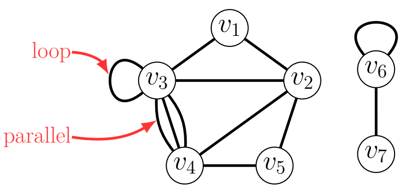




Euler's Multigraph



Multigraph (NOT simple)

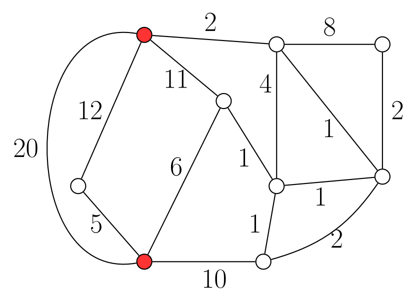


$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

$$E = \left\{ \begin{array}{l} (v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), \\ (v_2, v_5), (v_3, v_4), (v_3, v_4), (v_3, v_4), \\ (v_4, v_5), (v_6, v_7), (v_3, v_3), (v_6, v_6) \end{array} \right\}$$

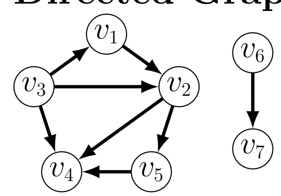
Handshaking Theorem still valid.

Weighted



How quickly can one route between the red ISPs?

Directed Graphs



$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

$$E = \left\{ \begin{array}{l} (v_1 \to v_2), (v_3 \to v_1), (v_3 \to v_2), (v_2 \to v_4), \\ (v_2 \to v_5), (v_3 \to v_4), (v_5 \to v_4), (v_6 \to v_7) \end{array} \right\}.$$

Ancestry graphs, tournaments, one-way streets, partialy ordered sets (Example 11.6), . . .

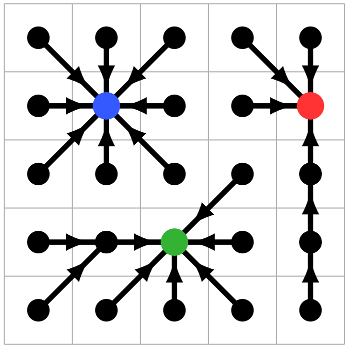
Problem Solving with Graphs

Graphs are everywhere because relationships are everywhere.

On the right is elevation data in a park.
One unit of rain falls on each grid-square.
Water flows to a neighbor of lowest elevation (e.g. $17 \to 1$)
Where should we install drains and what should their capacities be?

3	2	17	11	12
4	1	18	10	7
21	22	23	16	8
20	13	5	19	9
25	24	6	14	15

Model the problem as a directed graph.
Directed edges indicate how water flows: three disjoint trees.
The red, green and blue vertices are "sinks" (no out-going arrow).



Place drains at the sinks.
Drain capacities: blue=9 units, red=7 units and green=9 units.

The solution pops out once we formulate the problem as a graph.