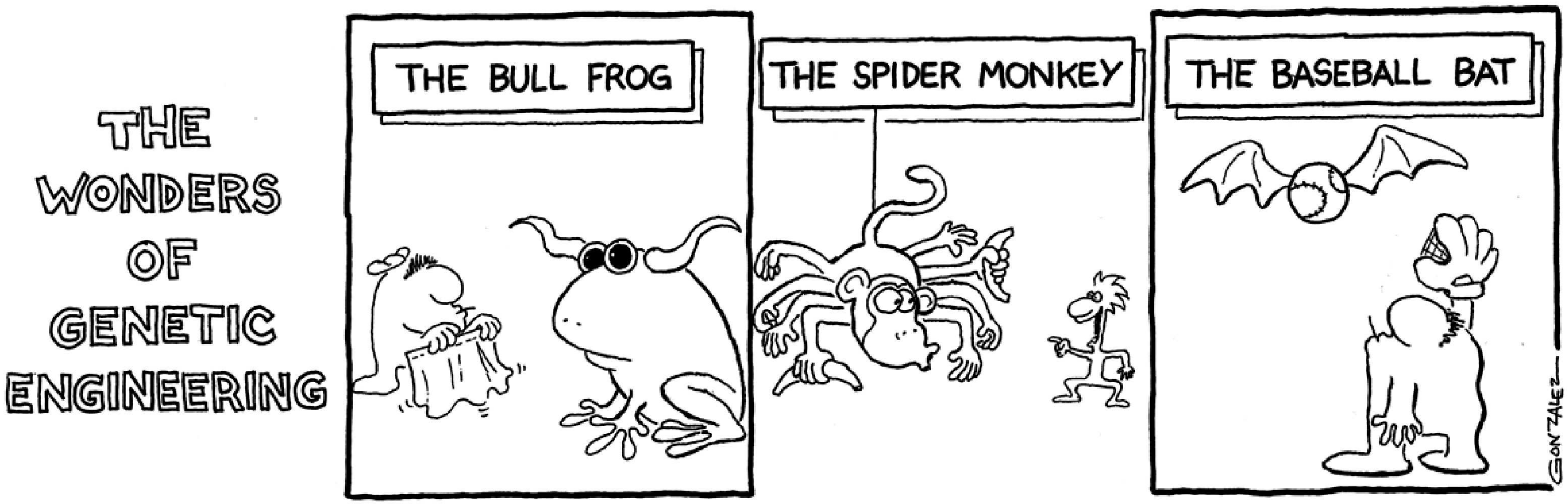
Foundations of Computer Science
Lecture 14

Advanced Counting

Sequences with Repetition
Union of Overlapping Sets: Inclusion-Exclusion
Pigeonhole Principle



Last Time

To count complex objects, construct a sequence of "instructions" that can be used to construct the object uniquely. The number of possible *sequences* of instructions equals the number of possible complex objects.

1. Sum and product Rules.
2. Build-up counting: $\binom{n}{k}$, n-bit sequences with k 1's; goody-bags.
3. Counting one set by counting another: bijection.
4. Permutations and combinations.
5. Binomial Theorem.

Today: Advanced Counting

1. Sequences with repetition.
   - Anagrams.
2. Inclusion-exclusion: extending the sum-rule to overlapping sets.
   - Derangements.
3. Pigeonhole principle.
   - Social twins.
   - Subset sums.

Selecting k from n Distinguishible Objects

	no repetition	with repetition
k-sequence		
k-subset		

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k-sequence	$\frac{n!}{(n-k)!}$	
k-subset		

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$(k_1, k_2, \cdots, k_r)$-sequence		

$(5, 4, 3)$-sequence of 5●, 4●, 3●

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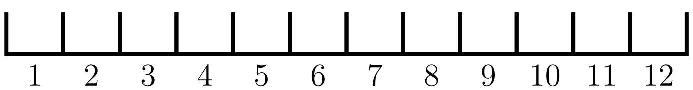

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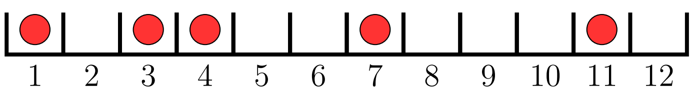
Choose slots for ●: $\binom{12}{5}$ ways



subset of slots used for each type

type - ●

$\{1, 3, 4, 7, 11\}$

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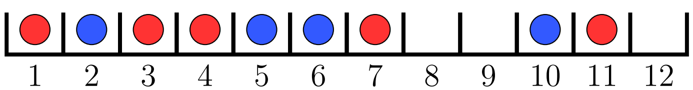

Choose slots for ●: $\binom{12}{5}$ ways

Then choose slots for ●: $\binom{7}{4}$ ways

subset of slots used for each type

type - ●	type - ●
$\{1, 3, 4, 7, 11\}$	$\{2, 5, 6, 10\}$

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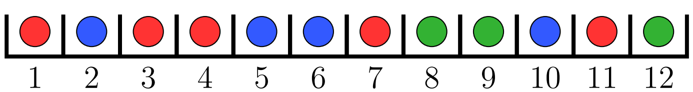

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$\{1,3,4,7,11\}$	$\{2,5,6,10\}$	$\{8,9,12\}$

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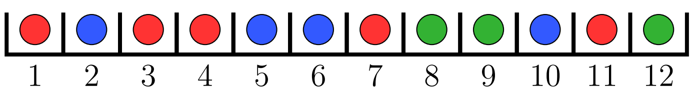

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Choose slots for ●: $\binom{12}{5}$ ways

Then choose slots for ●: $\binom{7}{4}$ ways

Then choose slots for ●: $\binom{3}{3}$ ways

Product rule:

$$\binom{12}{5, 4, 3} = \binom{12}{5} \times \binom{7}{4} \times \binom{3}{3}$$

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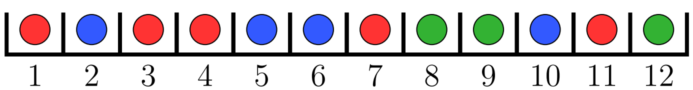

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$$= \frac{12!}{5! \cdot \cancel{7!}} \times \frac{\cancel{7!}}{4! \cdot \cancel{3!}} \times \frac{\cancel{3!}}{3! \cdot 0!}$$

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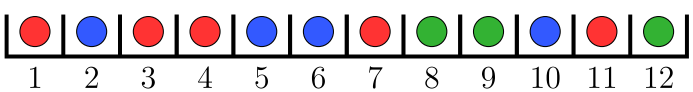
$(5, 4, 3)$-sequence of 5 red, 4 blue, 3 green



subset of slots used for each type

type - red	type - blue	type - green
$\{1, 3, 4, 7, 11\}$	$\{2, 5, 6, 10\}$	$\{8, 9, 12\}$

Choose slots for red: $\binom{12}{5}$ ways

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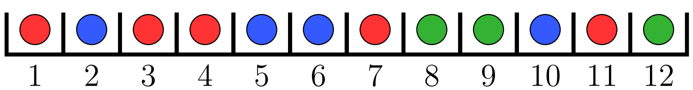
$(k_1, k_2, \cdots, k_r)$-sequence		$\binom{k_1+\cdots+k_r}{k_1, k_2, \ldots, k_r} = \dfrac{k!}{k_1! \cdot k_2! \cdots k_r!}$

$(5,4,3)$-sequence of 5🔴, 4🔵, 3🟢



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Anagrams: All "Words" Using the Letters of AARDVARK

A sequence of 8 letters: 3A's, 2R's, 1D, 1V, 1K.

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Number of such sequences is

$$\binom{8}{3,2,1,1,1} = \frac{8!}{3!\cdot 2!\cdot 1!\cdot 1!\cdot 1!} = 3360.$$

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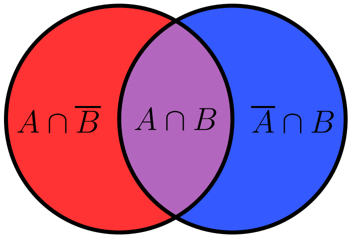
Exercise. What is the coefficient of $x^2y^3z^4$ in the expansion of $(x+y+z)^9$?
[Hint: Sequences of length 9 (why?) with 2 x's, 3 y's and 4 z's.]

Extending the Sum Rule to Overlapping Sets



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(Breaks $A \cup B$ into smaller sets.)

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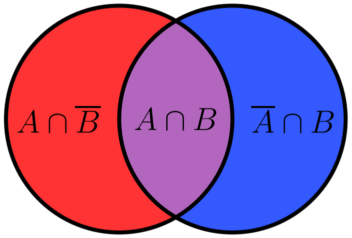

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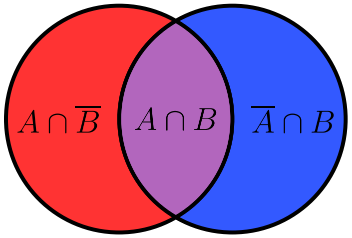

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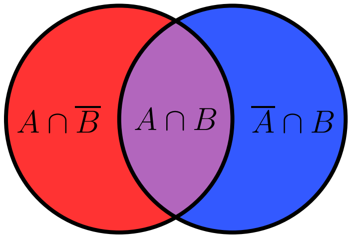

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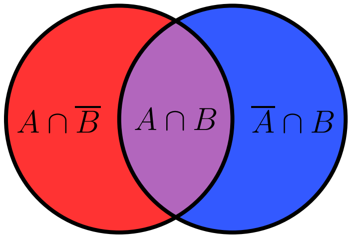

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$A \cap B = \{\text{numbers divisible by } 2 \text{ AND } 5\}$.	$\|A \cap B\| = 1$.	$(\|A \cap B\| = \lfloor 10/\text{lcm}(2,5) \rfloor)$

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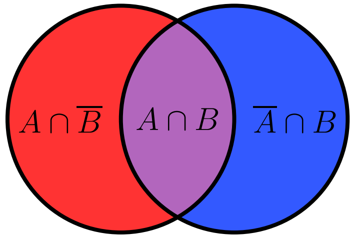

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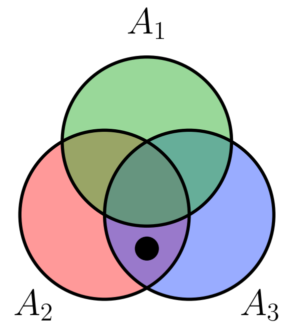
$\boldsymbol{A \cup B = \{\textbf{numbers divisible by 2 or 5}\}}$.

$$|A \cup B| = |A| + |B| - |A \cap B| = 5 + 2 - 1 = 6.$$

Inclusion-Exclusion

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Proof. ■



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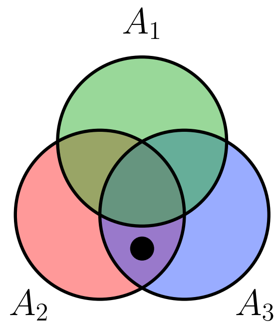
Proof. Consider $x \in A_2 \cap A_3$. How many times is x counted? ■



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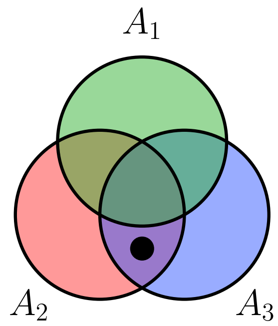
$\lvert A_1\rvert$	+	$\lvert A_2\rvert$	+	$\lvert A_3\rvert$	$-$	$\lvert A_1 \cap A_2\rvert$	$-$	$\lvert A_1 \cap A_3\rvert$	$-$	$\lvert A_2 \cap A_3\rvert$	+	$\lvert A_1 \cap A_2 \cap A_3\rvert$
0		+1		+1		0		0		-1		0

■



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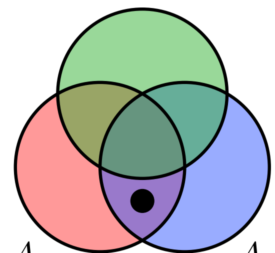

$\|A_1\|$	$+$	$\|A_2\|$	$+$	$\|A_3\|$	$-$	$\|A_1 \cap A_2\|$	$-$	$\|A_1 \cap A_3\|$	$-$	$\|A_2 \cap A_3\|$	$+$	$\|A_1 \cap A_2 \cap A_3\|$
0		+1		+1		0		0		−1		0

Contribution of x to sum is $+1$. Repeat for each region. ■

Inclusion-Exclusion

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

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$\lvert A_1\rvert$	+	$\lvert A_2\rvert$	+	$\lvert A_3\rvert$	−	$\lvert A_1 \cap A_2\rvert$	−	$\lvert A_1 \cap A_3\rvert$	−	$\lvert A_2 \cap A_3\rvert$	+	$\lvert A_1 \cap A_2 \cap A_3\rvert$
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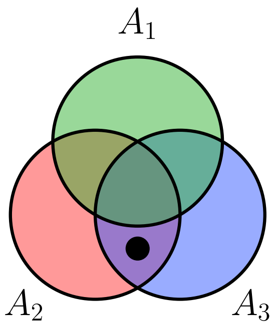

Example (Derangements). Give 3 coats to 3 girls so that noone gets their coat. How many ways?

Inclusion-Exclusion

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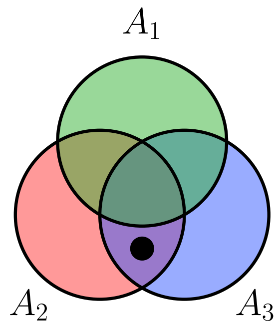
Proof. Consider $x \in A_2 \cap A_3$. How many times is x counted?



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Example (Derangements). Give 3 coats to 3 girls so that noone gets their coat. How many ways?

$A_i = \{\text{girl } i \text{ gets her coat}\}$. $|A_i| = 2!$.

$A_{ij} = \{\text{girls } i \text{ and } j \text{ get their coats}\}$. $|A_{ij}| = 1!$.

$A_{123} = \{\text{girls } 1, 2 \text{ and } 3 \text{ get their coats}\}$. $|A_{123}| = 1$.

Inclusion-Exclusion

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Proof. Consider $x \in A_2 \cap A_3$. How many times is x counted?

$\lvert A_1\rvert$	$+$	$\lvert A_2\rvert$	$+$	$\lvert A_3\rvert$	$-$	$\lvert A_1 \cap A_2\rvert$	$-$	$\lvert A_1 \cap A_3\rvert$	$-$	$\lvert A_2 \cap A_3\rvert$	$+$	$\lvert A_1 \cap A_2 \cap A_3\rvert$
0		+1		+1		0		0		−1		0

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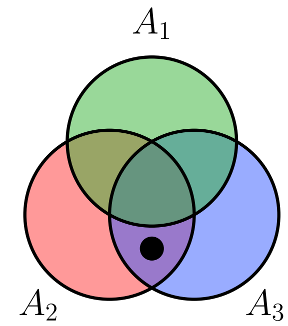

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$$\begin{aligned}|A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_{12}| - |A_{13}| - |A_{23}| + |A_{123}| \\ &= 2 + 2 + 2 - 1 - 1 - 1 + 1 = 4.\end{aligned}$$

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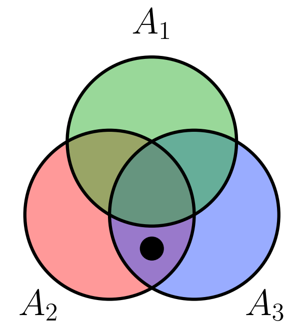
$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Proof. Consider $x \in A_2 \cap A_3$. How many times is x counted?

$\lvert A_1\rvert$	+	$\lvert A_2\rvert$	+	$\lvert A_3\rvert$	−	$\lvert A_1 \cap A_2\rvert$	−	$\lvert A_1 \cap A_3\rvert$	−	$\lvert A_2 \cap A_3\rvert$	+	$\lvert A_1 \cap A_2 \cap A_3\rvert$
0		+1		+1		0		0		−1		0

Contribution of x to sum is $+1$. Repeat for each region. ■



Example (Derangements). Give 3 coats to 3 girls so that noone gets their coat. How many ways?

$A_i = \{\text{girl } i \text{ gets her coat}\}$. $|A_i| = 2!$.

$A_{ij} = \{\text{girls } i \text{ and } j \text{ get their coats}\}$. $|A_{ij}| = 1!$.

$A_{123} = \{\text{girls } 1, 2 \text{ and } 3 \text{ get their coats}\}$. $|A_{123}| = 1$.

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_{12}| - |A_{13}| - |A_{23}| + |A_{123}| \\ &= 2 + 2 + 2 - 1 - 1 - 1 + 1 = 4. \end{aligned}$$

The answer we seek is $3! - 4 = 2$. (why?)

Exercise. How many numbers in 1,. . . ,100 are divisible by 2,3 or 5?

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If you have more guests than spare rooms, then some guests will have to share.

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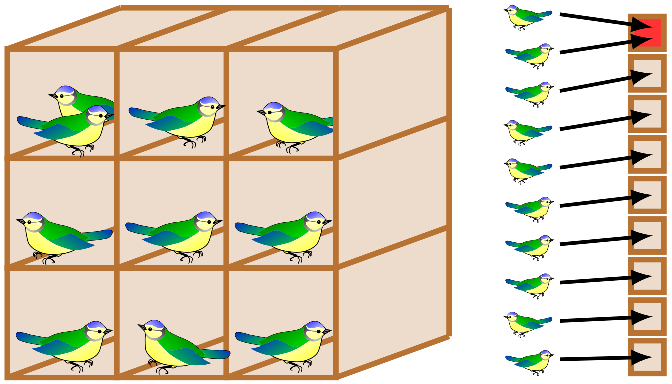

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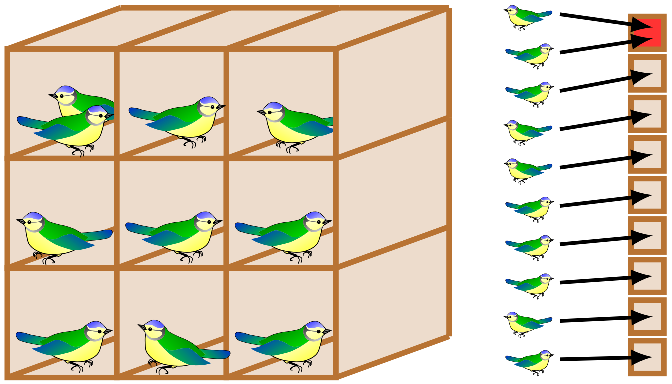

Proof. (By contraposition). Suppose no pigeonhole has 2 or more pigeons.
Let x_i be the number of pigeons in hole i, $x_i \leq 1$.

$$\text{number of pigeons} \ = \ \sum_i x_i \ \leq \ \sum_i 1 \ = \ \text{number of pigeonholes.} \qquad \blacksquare$$

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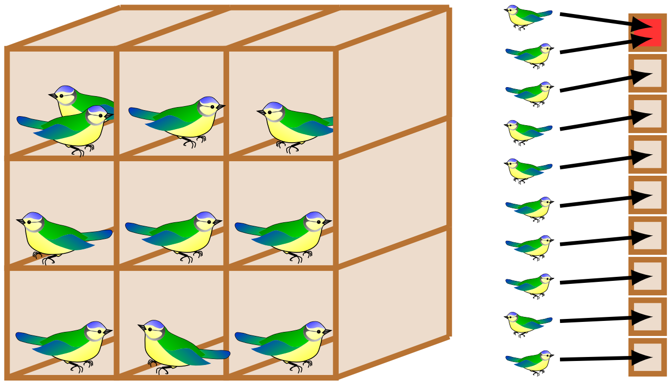

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We have 8 pigeons (the people) and 7 pigeonholes (the days of the week).

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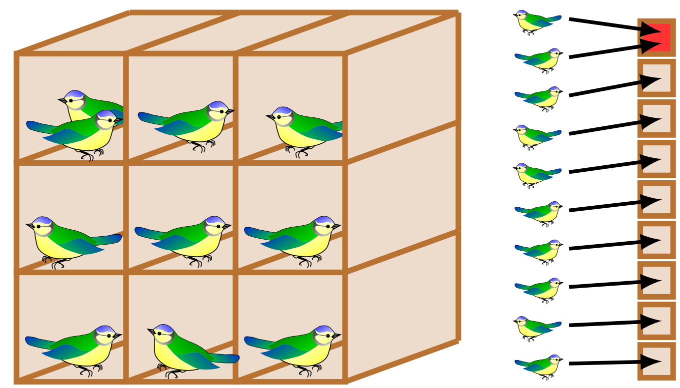

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How many people do you need to ensure two are born on a Monday?

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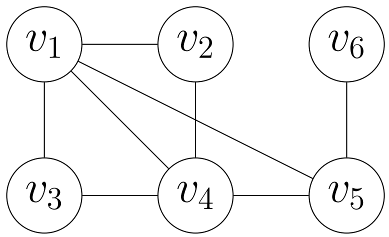

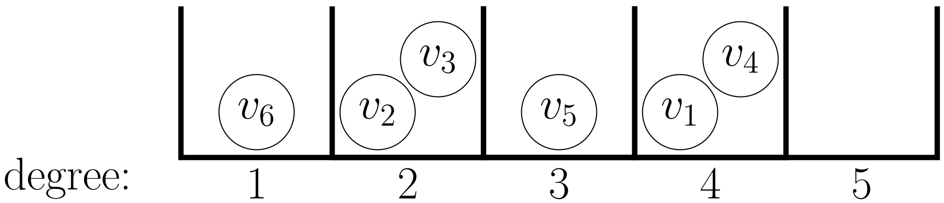

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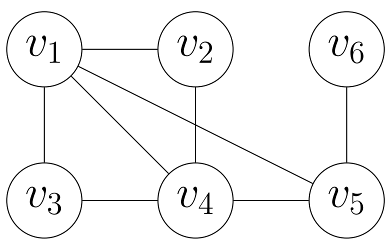

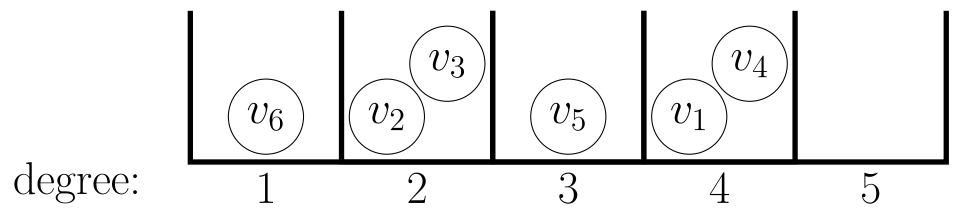

Degrees $1, 2, \ldots, (n-1)$, the pigeonholes. (Why no degree 0?)
Vertices $v_1, v_2, \ldots, v_n$, the pigeons.

n pigeons and $(n-1)$ pigeonholes, so at least two vertices are in the same degree-bin.

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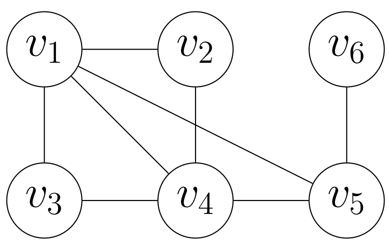

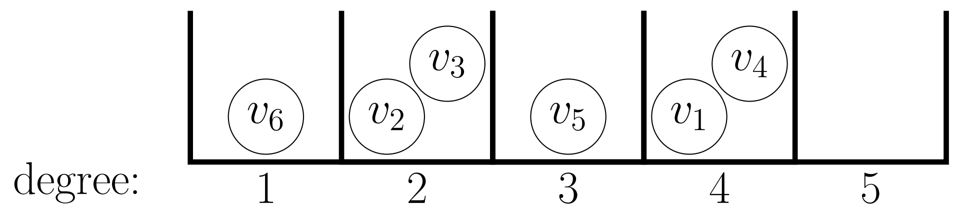

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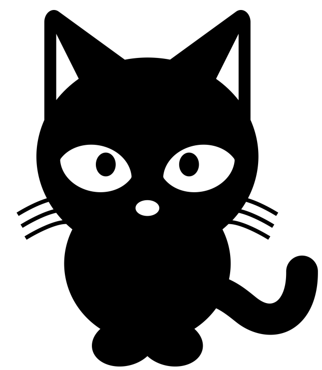
If the graph is not connected, no one has degree $n-1$.

Non-constructive proof: Who are those social twins? What are their degrees?

Non-Constructive Proof and the Eye-Spy Dilemma



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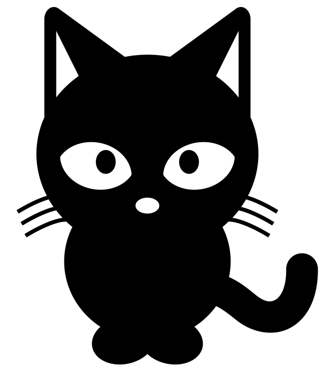

Prove to the 4 year old that the target exists in the picture

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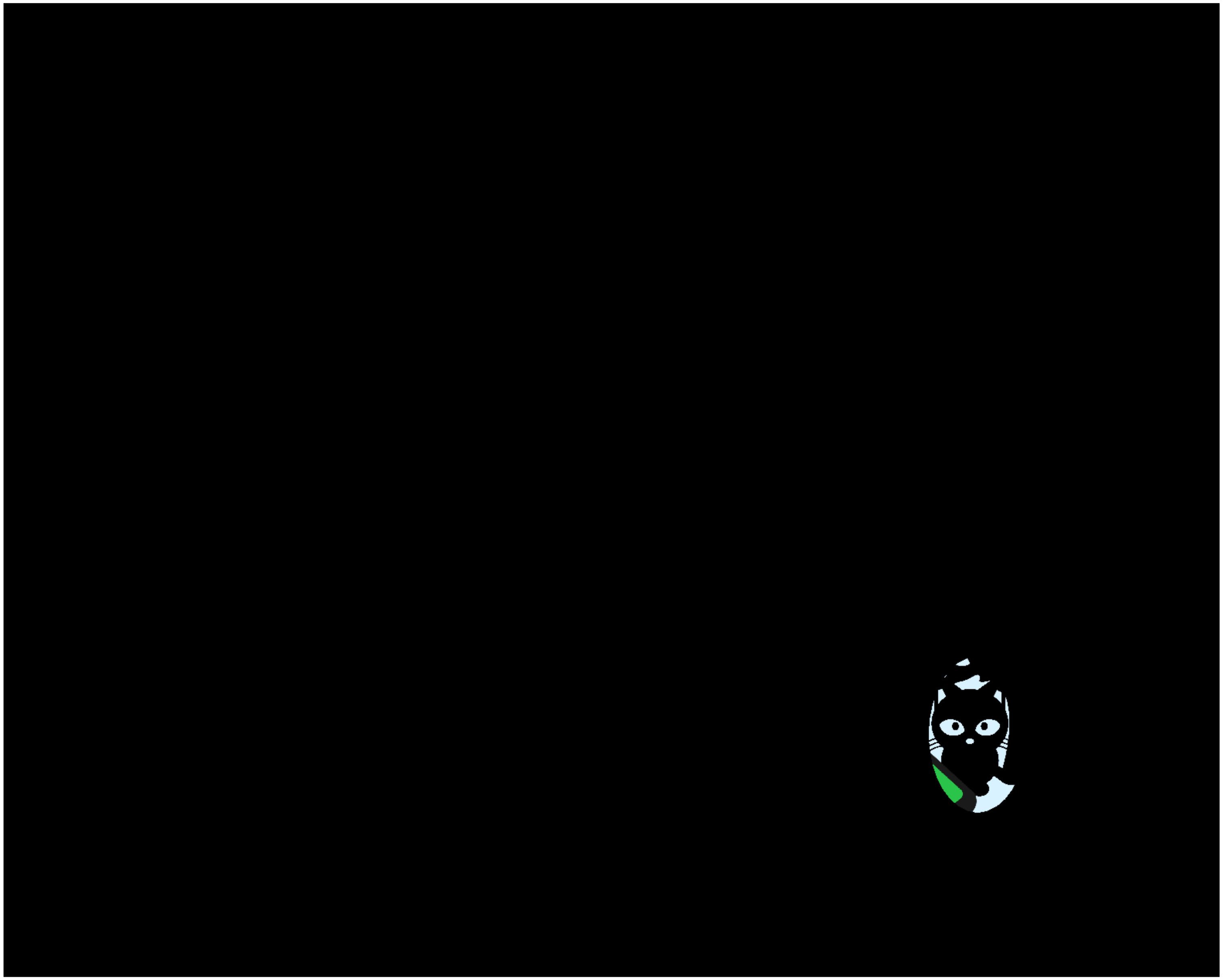

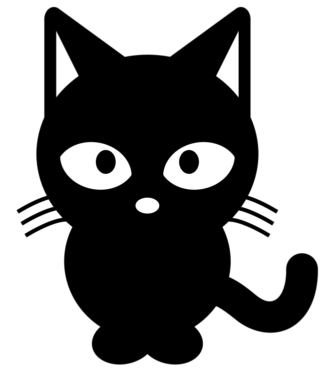

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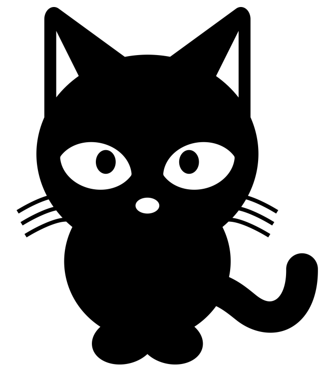

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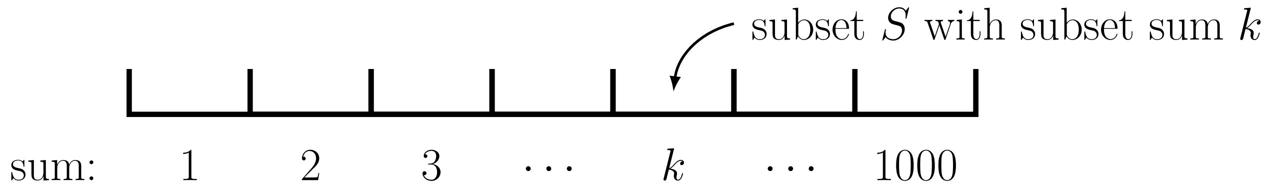

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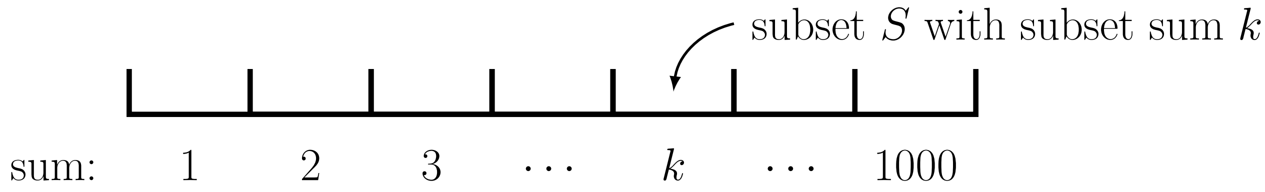

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At least two subsets must be in the same subset-sum-bin.

Practice. Exercise 14.6. $100 matching subsets problem has a solution. Professor didn't set a wild-goose chase.

Practice. Exercise 14.7.