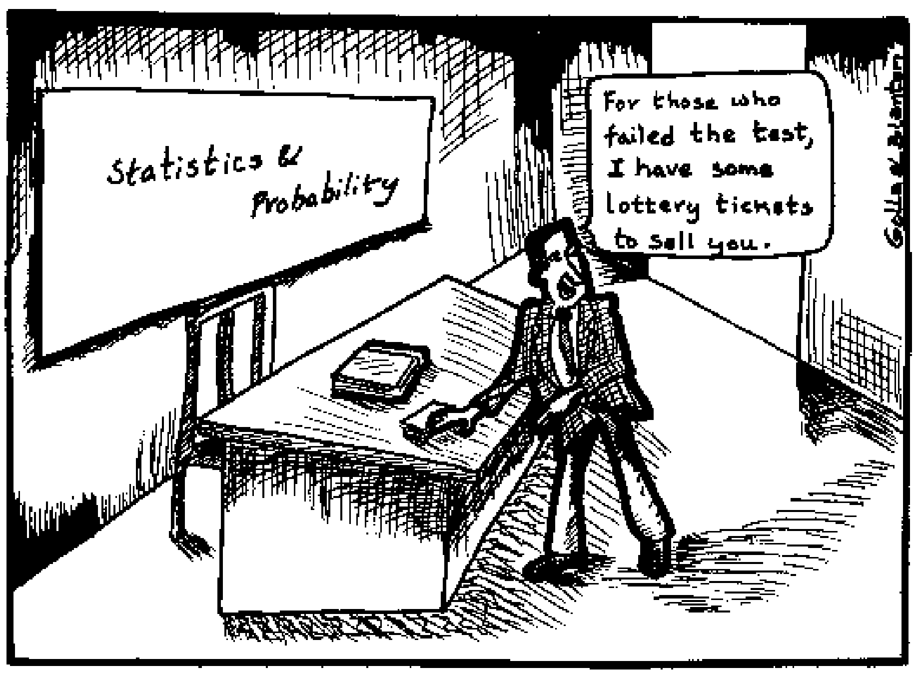
Foundations of Computer Science
Lecture 15

Probability

Computing Probabilities
Probability and Sets: Probability Space
Uniform Probability Spaces
Infinite Probability Spaces



The probable is what usually happens – Aristotle

Last Time

To count complex objects, construct a sequence of "instructions" that can be used to construct the object uniquely. The number of possible *sequences* of instructions equals the number of possible complex objects.

1. Counting
   - Sequences with and without repetition.
   - Subsets with and without repetition.
   - Sequences with specified numbers of each type of object: anagrams.

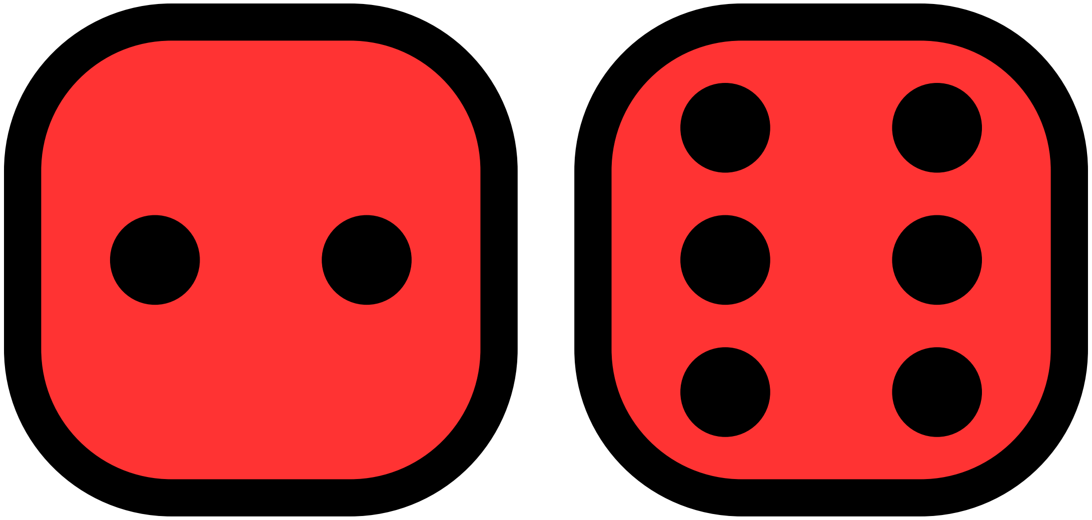
2. Inclusion-Exclusion (advanced technique).

3. Pigeonhole principle (simple but IMPORTANT technique).

Today: Probability

1. Computing probabilities.
   - Outcome tree.
   - Event of interest.
   - Examples with dice.
2. Probability and sets.
   - The probability space.
3. Uniform probability spaces.
4. Infinite probability spaces.

Probability



The Chance of Rain Tomorrow is 40%

What does the title mean? Either it will rain tomorrow or it won't.

The chances are 50% that a *fair* coin-flip will be H.

Flip 100 times. Approximately 50 will be H ← *frequentist* view.

1. You toss a *fair* coin 3 times. How many heads will you get?
2. You keep tossing a *fair* coin until you get a head. How many tosses will you make?

There's no answer. The outcome is uncertain. Probability handles such settings.

Birth of Mathematical Probability.

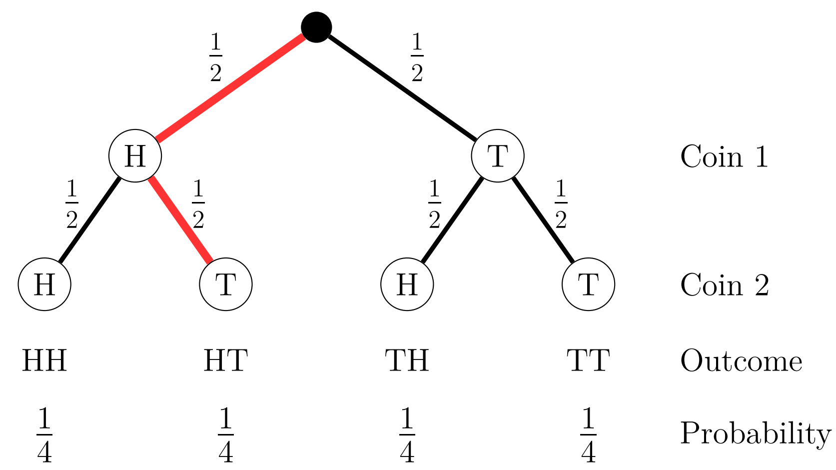
Antoine Gombaud,: Should I bet even money on at least one 'double-6' in 24 rolls of two dice?
Chevalier de Méré What about at least one 6 in 4 rolls of one die?

Blaise Pascal: Interesting question. Let's bring *Pierre de Fermat* into the conversation.

…a correspondence is ignited between these two mathematical giants

Toss Two Coins: You Win if the Coins Match (HH or TT)

1. You are analyzing an "experiment" whose outcome is uncertain.
2. **Outcomes.** Identify all *possible* outcomes using a *tree* of outcome *sequences*.
3. **Edge probabilities.** If one of k edges (options) from a vertex is chosen *randomly* then each edge has edge-probability $\frac{1}{k}$.
4. **Outcome-probability.** Multiply edge-probabilities to get outcome-probabilities.



Event of Interest

Toss two coins: you win if the coins match (HH or TT)

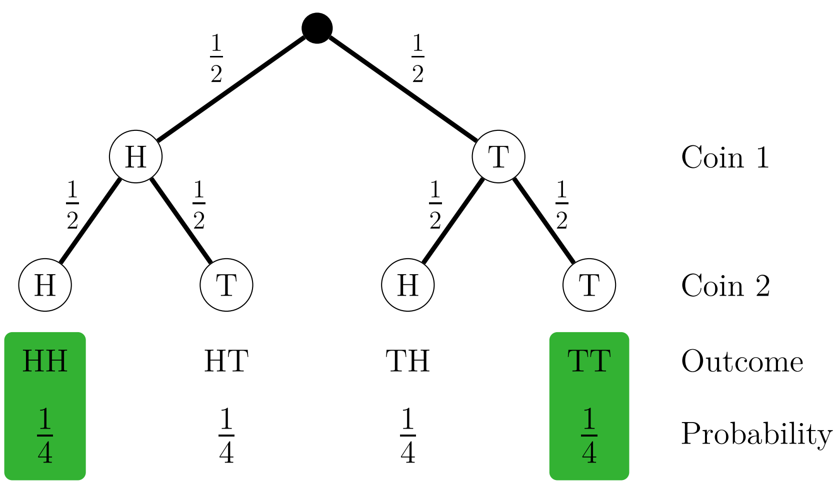
Question: When do you win?

Event: *Subset* of outcomes where you win.

5. **Event of interest.** Subset of the outcomes where you win.

6. **Event-probability.** Sum of its outcome-probabilities.

$$\text{event-probability} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$



Probability that you win is $\frac{1}{2}$, written as $\mathbb{P}[\text{“YouWin”}] = \frac{1}{2}$.

Go and do this experiment at home. Toss two coins 1000 times and see how often you win.

The Outcome-Tree Method

Become familiar with this 6-step process for analyzing a probabilistic experiment.

1. You are analyzing an experiment whose outcome is uncertain.
2. **Outcomes.** Identify *all possible* outcomes, the tree of *outcome sequences*.
3. **Edge-Probability.** Each edge in the outcome-tree gets a probability.
4. **Outcome-Probability.** Multiply edge-probabilities to get outcome-probabilities.
5. **Event of Interest $\mathcal{E}$.** Determine the subset of the outcomes you care about.
6. **Event-Probability.** The sum of outcome-probabilities in the subset you care about.

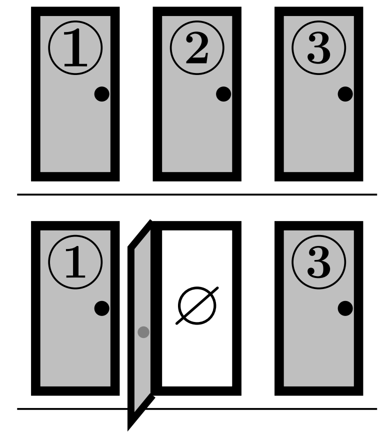
$$\mathbb{P}[\mathcal{E}] = \sum_{\text{outcomes } \omega \in \mathcal{E}} P(\omega).$$

$\mathbb{P}[\mathcal{E}] \sim$ frequency an outcome you want occurs over many repeated experiments.

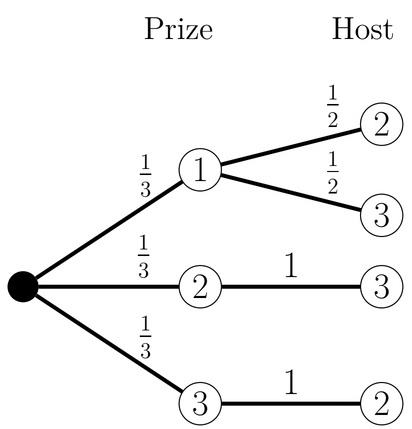
Pop Quiz. Roll two dice. Compute $\mathbb{P}[\text{first roll is less than the second}]$.

Let's Make a Deal: The Monty Hall Problem

1: Contestant at door 1.
2: Prize placed behind *random* door.
3: Monty opens empty door (*randomly* if there's an option).



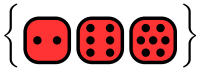
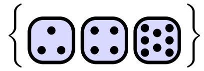
- Outcome-tree and edge-probabilities.
- Outcome-probabilities.
- Event of interest: "WinBySwitching".
- Event probability.



Outcome	Probability
$(1,2)$	$P(1,2)=\frac{1}{6}$
$(1,3)$	$P(1,3)=\frac{1}{6}$
$(2,3)$	$P(2,3)=\frac{1}{3}$
$(3,2)$	$P(3,2)=\frac{1}{3}$

$$\frac{1}{3}+\frac{1}{3}=\frac{2}{3}=\mathbb{P}[\text{“WinBySwitching”}]$$

Non-Transitive 3-Sided-Dice

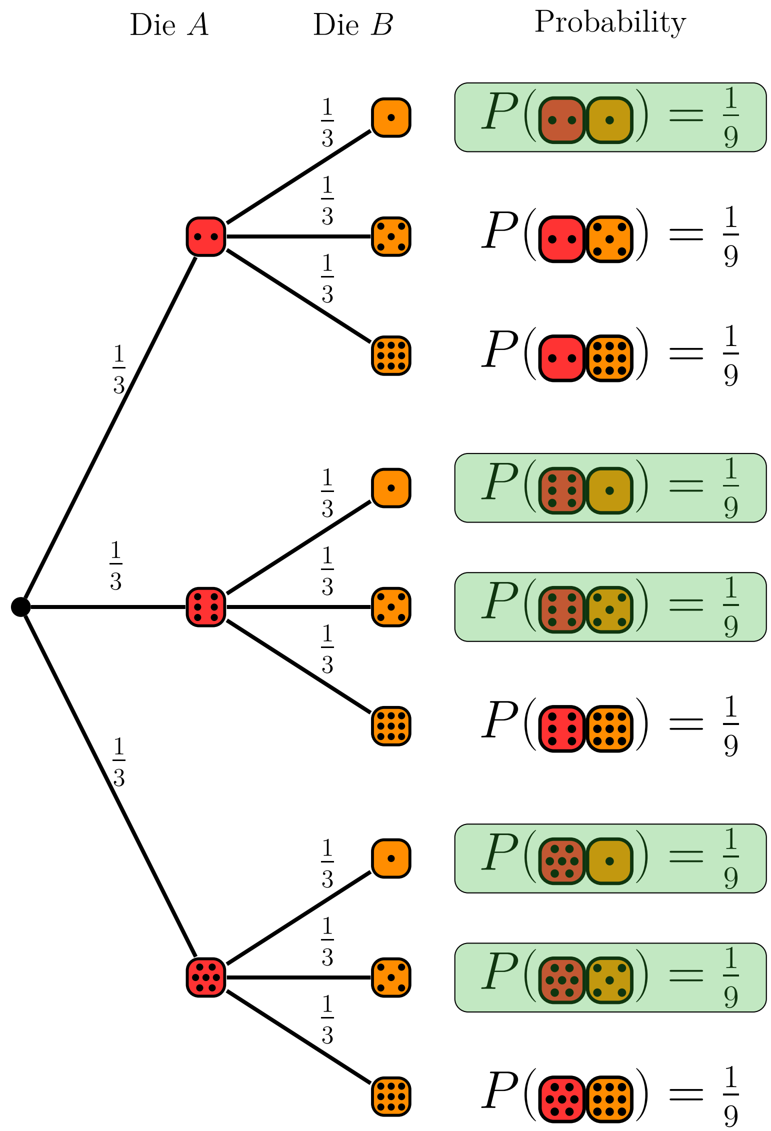
$\underline{A}$:  $\underline{B}$:  $\underline{C}$: 

(Dice from course 6.042J, ocw.mit.edu. See also Wikipedia, non-transitive dice.)

Your friend picks a die and then you pick a die.
E.g. friend picks B and then you pick A.

What is the probability that A beats B?

- Outcome-tree and outcome-probabilities.
- Uniform probabilities.
- Even of interest: outcomes where A wins.
- Number of outcomes where A wins: 5.
- $\mathbb{P}[A \text{ beats } B] = \frac{5}{9}$.



Conclusion: Die A beats Die B.

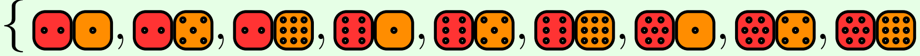
Pop Quiz. Compute $\mathbb{P}[B \text{ beats } C]$ and $\mathbb{P}[C \text{ beats } A]$ and show A beats B, B beats C and C beats A.

Probability and Sets: The Probability Space

1. **Sample Space** $\Omega = \{\omega_1, \omega_2, \ldots\}$, set of *possible* outcomes.
2. **Probability Function** $P(\cdot)$**.** Non-negative function $P(\omega)$, normalized to 1:

$$0 \leq P(\omega) \leq 1 \qquad \text{and} \qquad \sum_{\omega \in \Omega} P(\omega) = 1.$$

(Die A versus B)

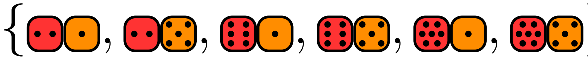
Ω {  }

$P(\omega)$: $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$

Events $\mathcal{E} \subseteq \Omega$ are subsets. Event probability $\mathbb{P}[\mathcal{E}]$ is the sum of outcome-probabilities.

"$A > B$" $\quad \mathcal{E}_1 =$ {  }

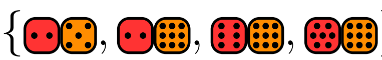
"Sum > 8" $\quad \mathcal{E}_2 =$ {  }

"$B < 9$" $\quad \mathcal{E}_3 =$ {  }

Combining events using logical connectors corresponds to set operations:

"$A > B$" $\vee$ "Sum > 8" $\quad \mathcal{E}_1 \cup \mathcal{E}_2 =$ {  }

"$A > B$" $\wedge$ "Sum > 8" $\quad \mathcal{E}_1 \cap \mathcal{E}_2 =$ {  }

$\neg$("$A > B$") $\quad \overline{\mathcal{E}_1} =$ {  }

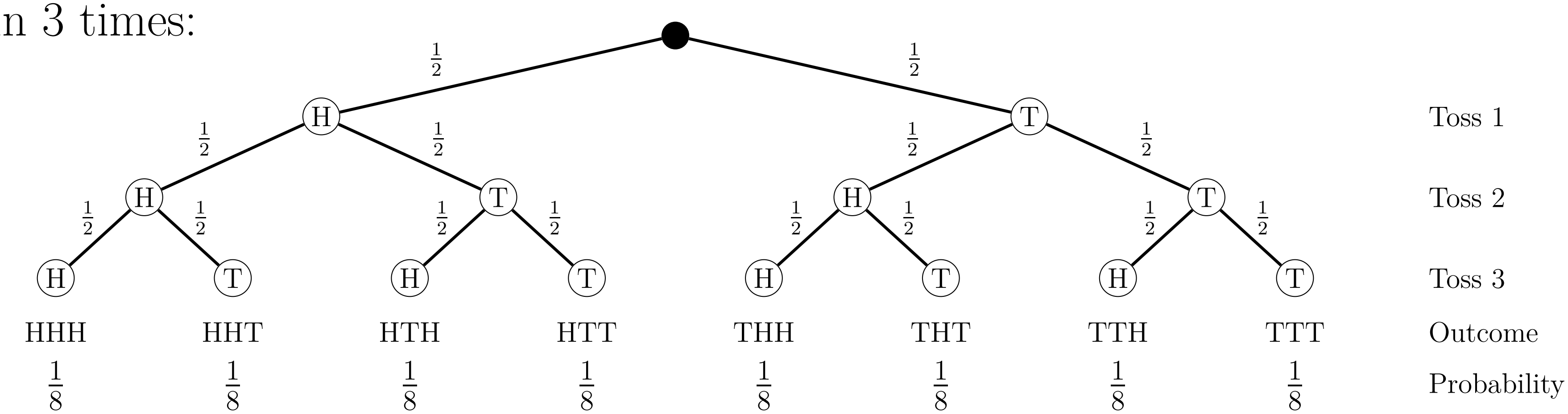
"$A > B$" $\rightarrow$ "$B < 9$" $\quad \mathcal{E}_1 \subseteq \mathcal{E}_3$

Important: Exercise 15.10. Sum rule, complement, inclusion-exclusion, union, implication and intersection bounds.

Uniform Probability Space : Probability ∼ Size

$$P(\omega) = \frac{1}{|\Omega|} \qquad \mathbb{P}[\mathcal{E}] = \frac{|\mathcal{E}|}{|\Omega|} = \frac{\text{number of outcomes in } \mathcal{E}}{\text{number of possible outcomes in } \Omega}.$$

Toss a coin 3 times:



$$\mathbb{P}[\text{“2 heads”}] = \frac{\text{number of sequences with 2 heads}}{\text{number of possible sequences in } \Omega} = \binom{3}{2} \times \frac{1}{8} = \frac{3}{8}.$$

Practice: Exercise 15.11.

1. You roll a pair of regular dice. What is the probability that the sum is 9?
2. You toss a fair coin ten times. What is the probability that you obtain 4 heads?
3. You roll die A ten times. Compute probabilities for: 4 sevens? 4 sevens AND 3 sixes? 4 sevens OR 3 sixes?

Poker: Probabilities of Full House and Flush

52 card deck has 4 suits (♠, ♥, ♢, ♣) and 13 ranks in a suit (A,K,Q,J,T,9,8,7,6,5,4,3,2).

Randomly deal 5-cards: each *set* of 5 cards is equally likely → uniform probability space.

$$\text{number of possible outcomes} = \binom{52}{5} \text{ possible hands.}$$

Full house: 3 cards of one rank and 2 of another. How many full-houses?
To construct a full house, specify $(\mathsf{rank}_3, \mathsf{suits}_3, \mathsf{rank}_2, \mathsf{suits}_2)$. Product rule:

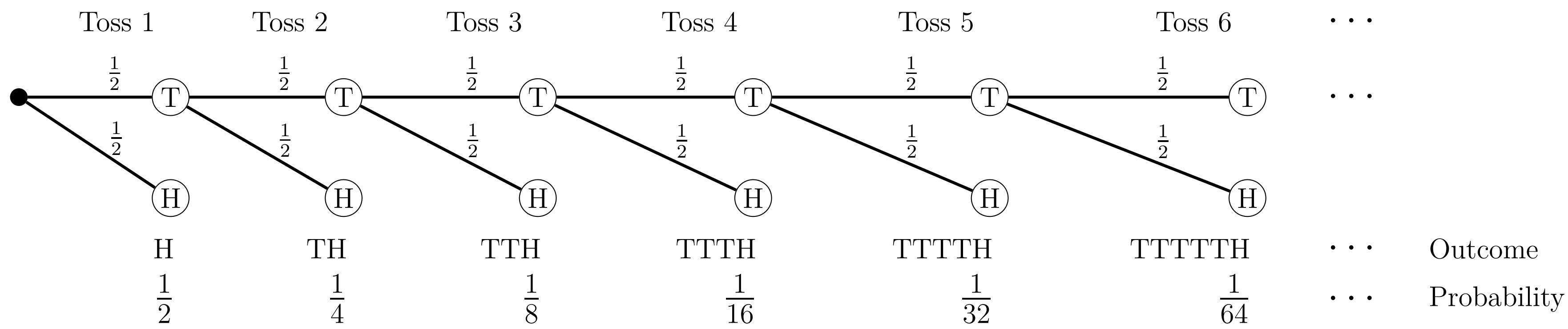
$$\#\text{ full houses} = 13\times\binom{4}{3}\times 12\times\binom{4}{2} \quad\rightarrow\quad \mathbb{P}[\text{“FullHouse”}] = \frac{13\times\binom{4}{3}\times 12\times\binom{4}{2}}{\binom{52}{5}} \approx 0.00144;$$

Flush: 5 cards of same suit. How many flushes?
To construct a flush, specify $(\mathsf{suit}, \mathsf{ranks})$. Product rule:

$$\#\text{ flushes} = 4\times\binom{13}{5} \quad\rightarrow\quad \mathbb{P}[\text{“Flush”}] = \frac{4\times\binom{13}{5}}{\binom{52}{5}} \approx 0.00198;$$

Full house is rarer. That's why full house beats flush.

Toss a Coin Until Heads: Infinite Probability Space

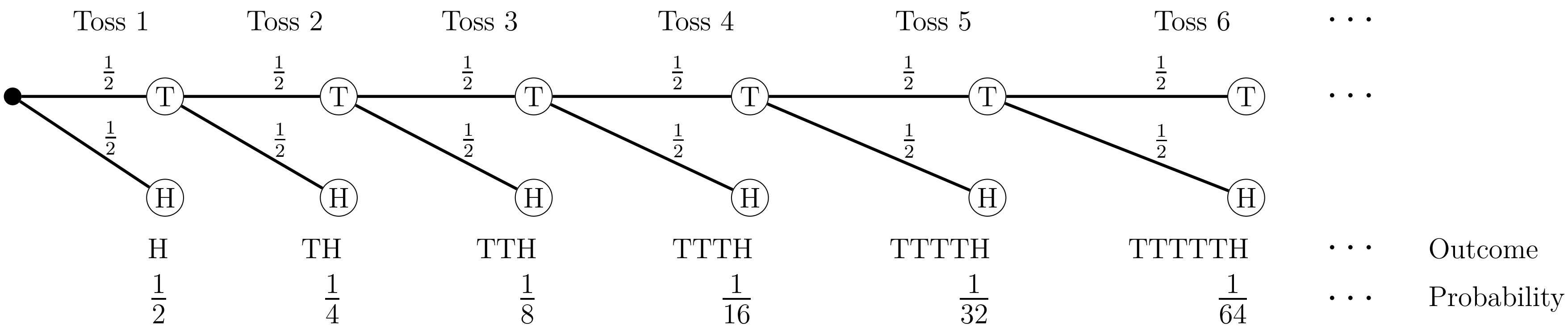


Ω	H	TH	$T^{\bullet 2}H$	$T^{\bullet 3}H$	$T^{\bullet 4}H$	$T^{\bullet 5}H$	$\cdots$	$T^{\bullet i}H$	$\cdots$
$P(\omega)$	$\frac{1}{2}$	$(\frac{1}{2})^2$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$(\frac{1}{2})^5$	$(\frac{1}{2})^6$	$\cdots$	$(\frac{1}{2})^{i+1}$	$\cdots$
# Tosses	1	2	3	4	5	6	$\cdots$	$i+1$	$\cdots$

Sum of outcome probabilities:

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \cdots = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1. \checkmark$$

Game: First Person To Toss H Wins. **Always Go First**



Ω	H	TH	$\mathrm{T}^{\bullet 2}\mathrm{H}$	$\mathrm{T}^{\bullet 3}\mathrm{H}$	$\mathrm{T}^{\bullet 4}\mathrm{H}$	$\mathrm{T}^{\bullet 5}\mathrm{H}$	$\cdots$	$\mathrm{T}^{\bullet i}\mathrm{H}$	$\cdots$
$P(\omega)$	$\frac{1}{2}$	$(\frac{1}{2})^2$	$(\frac{1}{2})^3$	$(\frac{1}{2})^4$	$(\frac{1}{2})^5$	$(\frac{1}{2})^6$	$\cdots$	$(\frac{1}{2})^{i+1}$	$\cdots$

The event "YouWin" is $\mathcal{E} = \{\mathrm{H}, \mathrm{T}^{\bullet 2}\mathrm{H}, \mathrm{T}^{\bullet 4}\mathrm{H}, \mathrm{T}^{\bullet 6}\mathrm{H}, \ldots\}$.

$$\mathbb{P}[\text{“YouWin”}] = \tfrac{1}{2} + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^5 + (\tfrac{1}{2})^7 + \cdots = \tfrac{1}{2}\sum_{i=0}^{\infty} (\tfrac{1}{4})^i = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \mathbf{\frac{2}{3}}.$$

Your odds improve by a factor of 2 if you go first (vs. second).