Foundations of Computer Science
Lecture 22

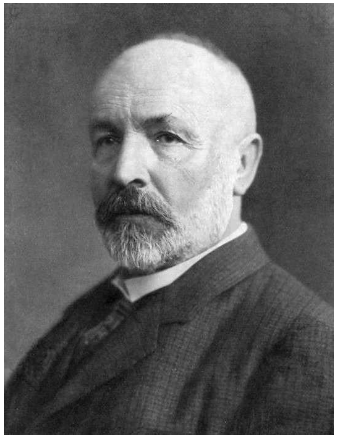
Infinity

Size versus Cardinality: Comparing "Sizes"
Countable: Sets Which Are Not "Larger" Than ℕ
Is There A Set "Larger" Than ℕ? Cantor's Diagonal Argument
Infinity and Computing

Our Short Stroll Through Discrete Math

1. Precise statements, proofs and logic.
2. **INDUCTION**.
3. Recursively defined structures and Induction. (Data structures; PL)
4. Sums and asymptotics. (Algorithm analysis)
5. Number theory. (Cryptography; probability; fun)
6. Graphs. (Relationships/conflicts; resource allocation; routing; scheduling,. . .)
7. Counting. (Enumeration and brute force algorithms)
8. Probability. (Real world algorithms involve randomness/uncertainty)
   - Inputs arrive in a random order;
   - Randomized algorithms (primality testing, machine learning, routing, conflict resolution . . .)
   - Expected value is a summary of what happens. Variance tells you how good the summary is.

Today: Infinity



Georg Cantor

1. Comparing "sizes" of sets: countable.
   - Rationals are countable.
2. Uncountable
   - Infinite binary strings.
3. What does Infinity have to do with computing?

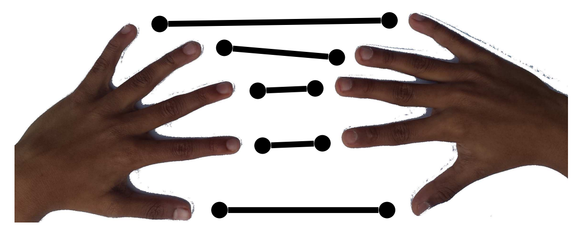
“Size” of a Set: Cardinality

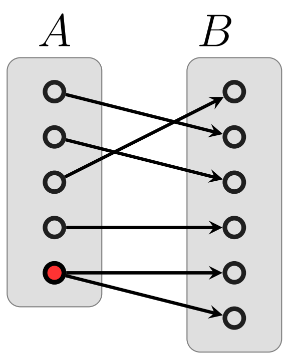
You have **5** fingers on each hand.

You must know how to count.

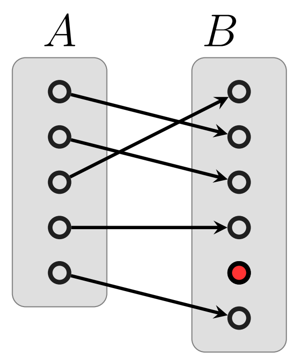
You have an *equal* number of fingers on each hand.

All you need is a correspondence.

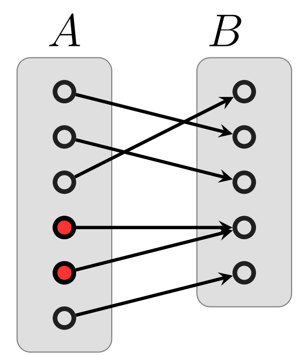




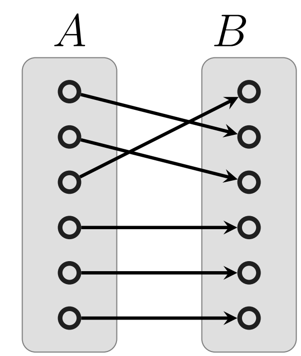
not a function



1-to-1;
(injection, $A \overset{\text{INJ}}{\mapsto} B$)
implies $|A| \leq |B|$



onto;
(surjection, $A \overset{\text{SUR}}{\mapsto} B$)
implies $|A| \geq |B|$



1-to-1 **and** onto
(bijection, $A \overset{\text{BIJ}}{\mapsto} B$)
implies $|A| = |B|$

Cardinality $|A|$(“size”), read “cardinality of A,” is the number of elements for finite sets

$|A| \leq |B|$ iff there is an injection (1-to-1) from A to B, i.e., $f : A \overset{\text{INJ}}{\mapsto} B$.
$|A| > |B|$ iff there is no injection from A to B.
$|A| \geq |B|$ iff there is an surjection (onto) from A to B, i.e., $f : A \overset{\text{SUR}}{\mapsto} B$.
$|A| = |B|$ iff there is an bijection (1-to-1 and onto) from A to B, i.e., $f : A \overset{\text{BIJ}}{\mapsto} B$.

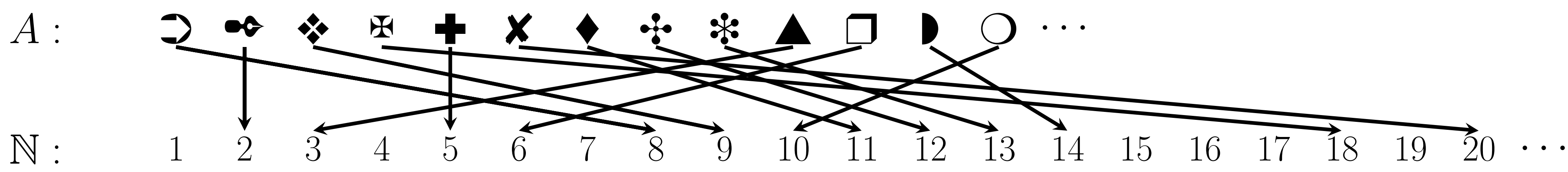
$$|A| \leq |B| \text{ AND } |B| \leq |A| \rightarrow |A| = |B|. \qquad \text{(Cantor-Bernstein Theorem)}$$

A Countable Set's Cardinality Is At Most $|\mathbb{N}|$

Finite sets : $|A| = n$ if and only if there is a bijection from A to $\{1, \ldots, n\}$.

Infinite sets: The set A is countable if $|A| \leq |\mathbb{N}|$. A is "smaller than" $\mathbb{N}$.

To show that A is countable you must find a 1-to-1 mapping from A to $\mathbb{N}$.



You cannot skip over any elements of A, but you might not use every element of $\mathbb{N}$.

<u>To prove that a function $f : A \mapsto \mathbb{N}$ is an injection:</u>

1: Assume f is *not* an injection. (Proof by contradiction.)
2: This means there is a pair $x, y \in A$ for which $x \neq y$ and $f(x) = f(y)$.
3: Use $f(x) = f(y)$ to prove that $x = y$, a contradiction. Hence, f *is* an injection.

All Finite Sets are Countable

$A = \{3, 6, 8\}$. To show $|A| \leq \mathbb{N}$, we give an injection from A to $\mathbb{N}$,

$$3 \mapsto 1 \qquad 6 \mapsto 2 \qquad 8 \mapsto 3.$$

For an arbitrary finite set $A = \{a_1, a_2, \ldots, a_n\}$, $\mathbb{N}$,

$$a_1 \mapsto 1 \qquad a_2 \mapsto 2 \qquad a_3 \mapsto 3 \qquad \cdots \qquad a_n \mapsto n.$$

Non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ are countable

How can this be? $\mathbb{N}_0$ contains every element in $\mathbb{N}$ *plus* 0?

To prove $|\,\mathbb{N}_0\,| \le |\,\mathbb{N}\,|$, we give an injection $f : \mathbb{N}_0 \overset{\text{INJ}}{\mapsto} \mathbb{N}$,

$$f(x) = x + 1, \qquad \text{for } x \in \mathbb{N}_0.$$

Proof. Assume f is not an injection. So, there are $x \ne y$ in $\mathbb{N}_0$ with $f(x) = f(y)$:

$$x + 1 \;=\; f(x) \;=\; f(y) \;=\; y + 1.$$

That is $x + 1 = y + 1$ or $x = y$, which contradicts $x \ne y$. ■

Also, $|\,\mathbb{N}\,| \le |\,\mathbb{N}_0\,|$ because $\mathbb{N} \subseteq \mathbb{N}_0 \rightarrow |\,\mathbb{N}_0\,| = |\,\mathbb{N}\,|$. (Cantor-Bernstein)

Bijection:

$\mathbb{N}_0$:	0	1	2	3	4	5	6	7	8	9	$\cdots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{N}$:	1	2	3	4	5	6	7	8	9	10	$\cdots$

Positive Even Numbers and Integers are Countable

$E = \{2, 4, 6, \ldots\}$. Surely $|E| = \frac{1}{2}|\,\mathbb{N}\,|$?

The bijection $f(x) = \frac{1}{2}x$ proves $|E| = |\,\mathbb{N}\,|$

E :	2	4	6	8	10	12	14	16	18	20	$\cdots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{N}$:	1	2	3	4	5	6	7	8	9	10	$\cdots$

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. $|\,\mathbb{Z}\,| = |\,\mathbb{N}\,|$.

	0		-1		-2		-3		-4		-5		-6		-7		-8		-9		-10	$\cdots$
	$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$	
$\mathbb{N}$:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	$\cdots$
		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		$\updownarrow$		
		1		2		3		4		5		6		7		8		9		10	$\cdots$	

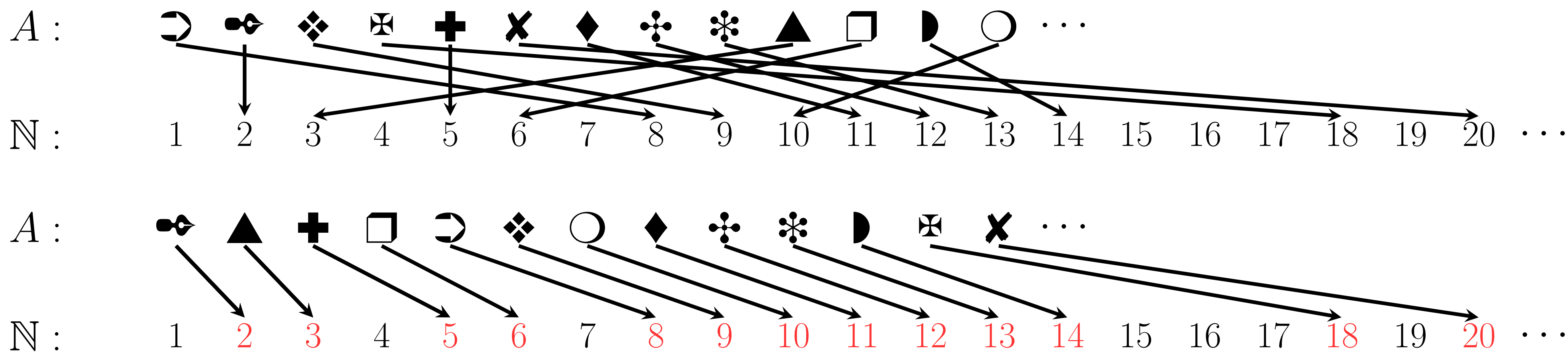
Exercise. What is a mathematical formula for the bijection?

Every Countable Set Can Be "Listed"

$\{3, 6, 8\}$ is a list. $E = \{2, 4, 6, \ldots\}$ is a list. What about $\mathbb{Z}$?

$$\cdots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots \qquad \leftarrow \text{not a list}$$

$$0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \cdots \qquad \leftarrow \text{list}$$



$\mathbb{N}_0 : \{0, 1, 2, 3, 4, 5, \ldots\}$ $\quad E : \{2, 4, 6, 8, 10, \ldots\}$ $\quad \mathbb{Z} : \{0, +1, -1, +2, -2, +3, -3, +4, -4, \ldots\}$

1. Different elements are assigned to different list-positions.
2. Can determine the list-position of *any* element in the set. For $\mathbb{Z}$,

$$\text{list position of } z = \begin{cases} 2z & z > 0; \\ 2|z| + 1 & z \leq 0; \end{cases}$$

Union of Two Countable Sets is Countable

A and B are countable, so they can be listed.

$$A = \{a_1, a_2, a_3, a_4, a_5, \ldots\} \qquad\qquad B = \{b_1, b_2, b_3, b_4, b_5, \ldots\}.$$

Here is a list for $A \cup B$

$$A \cup B = \{a_1, a_2, a_3, a_4, a_5, \ldots, b_1, b_2, b_3, b_4, b_5, \ldots\}. \qquad ✘$$

What is the list-position of b_1? Cannot use "…" twice.

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, \ldots\}.$$

list-position of a_i is $2i - 1$;
list-position of b_i is $2i$.

Pop Quiz. Get a list of $\mathbb{Z}$ with $A = \{0, -1, -2, -3, \ldots\}$ and $B = \{1, 2, 3, \ldots\}$ using union.

Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

This is surprising because between any two rationals there is another (not true for $\mathbb{N}$).

$\mathbb{Q}$ \ $\mathbb{Z}$	0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$\cdots$
$\mathbb{N}$ 1	$\frac{0}{1}$	$\frac{+1}{1}$	$\frac{-1}{1}$	$\frac{+2}{1}$	$\frac{-2}{1}$	$\frac{+3}{1}$	$\frac{-3}{1}$	$\frac{+4}{1}$	$\frac{-4}{1}$	$\cdots$
2	$\frac{0}{2}$	$\frac{+1}{2}$	$\frac{-1}{2}$	$\frac{+2}{2}$	$\frac{-2}{2}$	$\frac{+3}{2}$	$\frac{-3}{2}$	$\frac{+4}{2}$	$\frac{-4}{2}$	$\cdots$
3	$\frac{0}{3}$	$\frac{+1}{3}$	$\frac{-1}{3}$	$\frac{+2}{3}$	$\frac{-2}{3}$	$\frac{+3}{3}$	$\frac{-3}{3}$	$\frac{+4}{3}$	$\frac{-4}{3}$	$\cdots$
4	$\frac{0}{4}$	$\frac{+1}{4}$	$\frac{-1}{4}$	$\frac{+2}{4}$	$\frac{-2}{4}$	$\frac{+3}{4}$	$\frac{-3}{4}$	$\frac{+4}{4}$	$\frac{-4}{4}$	$\cdots$
5	$\frac{0}{5}$	$\frac{+1}{5}$	$\frac{-1}{5}$	$\frac{+2}{5}$	$\frac{-2}{5}$	$\frac{+3}{5}$	$\frac{-3}{5}$	$\frac{+4}{5}$	$\frac{-4}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Intuition suggests
$|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}| \gg |\mathbb{N}|$. ✘

$$\mathbb{Q} = \{\tfrac{0}{1}, \tfrac{+1}{1}, \tfrac{+1}{2}, \tfrac{0}{2}, \tfrac{0}{3}, \tfrac{+1}{3}, \tfrac{-1}{3}, \tfrac{-1}{2}, \tfrac{-1}{1}, \tfrac{+2}{1}, \tfrac{+2}{2}, \tfrac{+2}{3}, \tfrac{+2}{4}, \tfrac{-1}{4}, \tfrac{+1}{4}, \tfrac{0}{4}, \tfrac{0}{5}, \ldots\}$$

$|\{\text{Rational Values}\}| \le |\mathbb{Q}| \le |\mathbb{N}|.$

Exercise. What is a mathematical formula for the list-position of $z/n \in \mathbb{Q}$?

Programs are Countable

Programs are finite binary strings. We show that all finite binary strings $\mathcal{B}$ are countable.

$$\mathcal{B} = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, \ldots\} \qquad \leftarrow\text{list}$$

Pop Quiz. What is the list-position of 0110?

Exercise. For the $(k+1)$-bit string $b = b_k b_{k-1} \cdots b_1 b_0$, define the strings numerical value:

$$\text{value}(b) = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \cdots + b_{k-1} \cdot 2^{k-1} + b_k \cdot 2^k.$$

Show:

$$\text{list-position of } b \;=\; 2^{\text{length}(b)} + \text{value}(b).$$

$\mathbb{N}_0, E, \mathbb{Z}, \mathbb{Q}, \mathcal{B}$ are countable,... **Is Everything Countable?**

Infinite Binary Strings are Uncountable

Cantor's Diagonal Argument: Assume there is a list of *all* infinite binary strings.

$$
\begin{array}{r@{\;}l}
b_1\colon & \mathbf{0}\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\cdots \\
b_2\colon & 0\;\mathbf{0}\;1\;1\;0\;1\;0\;0\;1\;0\;0\;0\;0\;1\;0\cdots \\
b_3\colon & 1\;1\;\mathbf{0}\;0\;0\;0\;1\;0\;0\;0\;0\;1\;1\;0\;0\cdots \\
b_4\colon & 1\;0\;1\;\mathbf{0}\;0\;1\;0\;0\;0\;0\;1\;0\;0\;0\;0\cdots \\
b_5\colon & 0\;1\;1\;0\;\mathbf{1}\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\cdots \\
b_6\colon & 0\;1\;0\;1\;1\;\mathbf{0}\;0\;0\;0\;1\;0\;0\;0\;0\;0\cdots \\
b_7\colon & 0\;0\;1\;0\;0\;1\;\mathbf{0}\;0\;0\;0\;0\;0\;0\;0\;1\cdots \\
b_8\colon & 0\;0\;1\;0\;1\;1\;0\;\mathbf{1}\;0\;0\;0\;0\;1\;0\;0\cdots \\
b_9\colon & 0\;0\;0\;0\;1\;1\;0\;0\;\mathbf{0}\;0\;1\;0\;0\;0\;0\cdots \\
b_{10}\colon & 1\;0\;1\;1\;1\;0\;1\;0\;0\;\mathbf{1}\;1\;0\;0\;0\;0\cdots \\
\vdots &
\end{array}
$$

Consider the "diagonal string"

$$b = 0000100101\cdots$$

Flip the bits,

$$\bar{b} = 1111011010\cdots$$

$\bar{b}$ is not in the list (differs in the ith position from b_i), a contradiction.

Reals are Uncountable

Every real has an infinite binary representation and every infinite binary string evaluates to a real number.

$$\text{e.g.} \qquad 0.0011111111111111\cdots = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \cdots = \frac{1}{2}.$$

That is $|\{\text{reals in } [0,1]\}| = |\{\text{infinte binary stings}\}| > |\,\mathbb{N}\,|$.

Infinity and Computing

Cantor took on the abstract beast Infinity. (1874)
$\sim$ 60 years later, Alan Turing asked the abstract question: What can we compute? (1936)

Every binary function f on $\mathbb{N}$ corresponds to a infinite binary string $f(1)f(2)f(3)\cdots$,

n:	1	2	3	4	5	6	7	8	9	10	$\cdots$
$f(n)$:	0	1	1	0	1	0	0	0	1	1	$\cdots$

Every program is a finite binary string. For example,

```
int main();     //a program that does nothing
```

is the finite binary string (ASCII code)

0110100101101110011101000010000001101101011000010110100101101110001010000010100100111011

Programs $\leftarrow$ Countable
Functions $\leftarrow$ Uncountable $\quad\rightarrow\quad |\{\text{functions on } \mathbb{N}\}| \gg |\{\text{programs}\}|$

There are MANY MANY functions that cannot be computed by programs!
Are there interesting, useful functions that cannot be computed by programs?