Foundations of Computer Science Lecture 23

Languages: What is Computing?

A Formal Model of a Computing Problem
Decision Problems and Languages
Describing a Language: Regular Expressions
Complexity of a Computing Problem

"Say what's on your mind, Harris—the language of dance has always eluded me."

Last Time

1. Comparing infinite sets.

2. Countable.
   - $\mathbb{N}_0, E, \mathbb{Z}, \mathbb{Q}$ are countable.
   - Finite binary strings $\mathcal{B}$ is countable.

3. Uncountable
   - *Infinite* binary strings are uncountable.
   - Reals are uncountable.

4. Infinity and computing.
   - Programs are finite binary strings (countable).
   - Functions we might like to compute are infinite binary strings (uncountable).
   - Conclusion: there are **MANY** functions which *cannot* be computed by programs.

Today: Languages: What is Computing?

1. Decision problems.
2. Languages.
   - Describing a language.
3. Complexity of a computing problem.

What is a Computing Problem?

Decide YES or NO whether a given integer $n \in \mathbb{N}$ is prime.

List the primes in increasing order (primes are countable),

$$\text{primes} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots\}$$

Given $n \in \mathbb{N}$, walk through this list.

1: If you come to n output YES.
2: If you come to a number bigger than n, output NO.

Not the smartest approach to primality testing, but gets to the heart of computing

LANGUAGES

Decision Problems

$$\mathcal{L}_{\text{prime}} = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, \ldots\}.$$ (primes in binary)

9 is prime $\leftrightarrow$ the *string* 1001 is in $\mathcal{L}_{\text{prime}}$.

The light is off. Every push toggles between on and off.
Given the number of pushes, decide whether the light is on or off.
Encode number of pushes by a binary string, e.g. 101 means 5 pushes.

$$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}.$$

The light is on for 1010 pushes, if and only if $1010 \in \mathcal{L}_{\text{push}}$.

The door should open if a person is on the mat.
Walk on (1) or off (0). E.g. 10110 means on, off, on, on, off $\rightarrow$ open.

$$\mathcal{L}_{\text{door}} = \{1, 11, 101, 110, 111, 1011, 1101, 1110, 1111, \ldots\}.$$

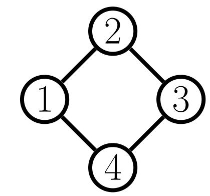
Given input w, e.g. $w = 1011$, the door is open if and only if $w \in \mathcal{L}_{\text{door}}$.

Decision problems can be formulated as testing membership in a set of strings

A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance.

"Is there a path of length at most 3 between nodes ① and ③ in the graph above."

becomes

$$\text{“ } 1, 2, 3, 4 \mid (1, 2)(2, 3)(3, 4)(4, 1) \mid 1, 3 \mid 3 \text{ ”}$$

nodes — edges — endpoints of path — target distance

The graph problem can be encoded as a binary string using ASCII

0011000100101100001100100010110000110011001011000011010001111100001010000011000100101100001100100010100100101000001100100010110000110011
00101001001010000011001100101100001101000010100100101000001101000010110000110001001010010111110000110001001011000011001101111110000110011.

$$\mathcal{L}_{\text{path}} = \left\{ \begin{matrix} \text{All strings of the form “nodes} \mid \text{edges} \mid \text{endpoints of path} \mid \text{target distance” for which} \\ \text{the distance between the endpoints in the graph is at most the target distance.} \end{matrix} \right\}$$

Pop Quiz. YES or NO: " $1, 2, 3, 4, 5 \mid (1, 2)(2, 3)(3, 5)(3, 4) \mid 1, 5 \mid 2$ "

Is Optimization Really Harder than Decision?

If you can solve the decision problem, you can solve the optimization problem.

Is there a path in the graph between nodes ⓧ and ⓨ of length at most **1**? NO
Is there a path in the graph between nodes ⓧ and ⓨ of length at most **2**? NO
Is there a path in the graph between nodes ⓧ and ⓨ of length at most **3**? NO
Is there a path in the graph between nodes ⓧ and ⓨ of length at most **4**? YES

You ask the decision question until the answer is YES.

The minimum-pathlength between ⓧ and ⓨ is 4.

It can take long, but it works.

Decision and optimization are "equivalent" when it comes to *solvability*.

A computing problem is a decision problem.

Languages

Standard formulation of a decision problem:

Problem: GRAPH-DISTANCE-D
Input: Finite graph G; nodes x, y; target distance D.
Question: Is there an (x,y)-path in G of length at most D.

Every decision problem has a YES-set, which we usually don't explicitly list.

$$\text{YES-set} = \{\text{input strings } w \text{ for which the answer is YES}\} = \{w_1, w_2, w_3, \ldots\}.$$

← A *language* is any set of finite binary strings

A computing problem is a YES-set, a set of *finite* binary strings.

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	$\leftarrow$ finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	$\leftarrow$ all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	
$\mathcal{L}_{\text{push}}$	$\{1, 01, 11, 001, 011, 101, 111, 0001, 0011, \ldots\}$	
$\mathcal{L}_{\text{door}}$	$\{1, 11, 101, 110, 111, 1011, 1101 \ldots\}$	
$\mathcal{L}_{\text{unary}}$	$\{\varepsilon, 1, 11, 111, 1111, \ldots\} = \{1^{\bullet n} \mid n \geq 0\}$	$\leftarrow$ strings of 1s
$\mathcal{L}_{(01)^n}$	$\{\varepsilon, 01, 0101, 010101, \ldots\} = \{(01)^{\bullet n} \mid n \geq 0\}$	
$\mathcal{L}_{0^n1^n}$	$\{01, 0011, 000111, \ldots\} = \{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$	
$\mathcal{L}_{\text{pal}}$	$\{\varepsilon, 0, 1, 00, 11, 000, 010, 101, 111, \ldots\}$	$\leftarrow$ palindromes
$\mathcal{L}_{\text{repeated}}$	$\{\varepsilon, 00, 11, 0000, 0101, 1010, 1111, \ldots\}$	$\leftarrow$ repeated strings

Describing a Language: String Patterns and Variables

An example where there is a clear pattern,

$$\mathcal{L} = \{\varepsilon, 01, 0101, 010101, \ldots\}.$$

Use a variable to formally define $\mathcal{L}$:

$$\mathcal{L} = \{w \mid w = (01)^{\bullet n},\ \text{where } n \geq 0\}.$$

(informally $\{(01)^{\bullet n} \mid n \geq 0\}$)

More than one variable:

$$\{u \bullet v \mid u \in \Sigma^* \text{ and } v = u^{\mathrm{R}}\} = \{\varepsilon, 00, 11, 0000, 0110, 1001, 1111, \ldots\}.$$

← even palindromes

Exercise. Define $\mathcal{L}_{\text{add}} = \{0100, 011000, 001000, 00110000, 00010000, 0001100000, 01110000, 0011100000, 000111000000, \ldots\}$

Ans: $\{0^{\bullet n} \bullet 1^{\bullet m} \bullet 0^{\bullet n+m}\}$

For more complicated patterns, we use regular expressions, e.g. the Unix/Linux command

ls FOCS$*$

(Lists everything that starts with FOCS ($*$ is the "wild-card").)

The Regular Expression: $\{1,11\} \bullet \overline{\{0,01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1,11\} \qquad \{0,01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement (Familiar.)
concatenation $\bullet$, Kleene-star * (What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\{0,01\} \bullet \{0,11\} = \{00,011,010,0111\}$$

$$\{0,11\} \bullet \{0,01\} = \{00,001,110,1101\} \qquad \mathcal{L}_1 \bullet \mathcal{L}_2 \neq \mathcal{L}_2 \bullet \mathcal{L}_1$$

$$\{0,01\} \bullet \{0,01\} = \{0,01\}^{\bullet 2} = \{00,001,010,0101\} \qquad \text{(self-concatenation)}$$

Pop Quiz. What is $\{0,01\} \bullet \{1,10\}$? What is $\{0,01\}^{\bullet 3}$? What is $\{0,01\}^{\bullet 0}$?

Kleene star: All possible concatenations of a finite number of strings from a language.

$$\{0,01\}^* = \{\varepsilon, 0, 01, 00, 001, 010, 0101, 000, 0001, 0010, \ldots\} = \bigcup_{n=0}^{\infty} \{0,01\}^{\bullet n};$$

$$\{1\}^* = \{\varepsilon, 1, 11, 111, 1111, 11111, \ldots\} = \bigcup_{n=0}^{\infty} \{1\}^{\bullet n}.$$

Pop Quiz. Which of the strings $\{101110, 00111, 00100, 01100\}$ can you generate using $\{0,01\}^* \bullet \{1,10\}^*$?

The Regular Expression: $\{1,11\} \bullet \overline{\{0,01\}^*} \bullet (\{00\} \cup \{1\}^*)$

$$\begin{aligned}\{0,01\}^* &= \{\varepsilon, 0, 01, 00, 001, 010, 0101, 000, 0010, \ldots\}\\ \{1\}^* &= \{\varepsilon, 1, 11, 111, 1111, 11111, \ldots\}\end{aligned}$$

To generate 1110111:

$$11 \in \{1,11\}$$

$$10 \in \overline{\{0,01\}^*}$$

$$111 \in \{00\} \cup \{1\}^*$$

Hence $1110111 \in \{1,11\} \bullet \overline{\{0,01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Pop Quiz Is there another way to generate 1110111?

Pop Quiz Yes or no: $11110010 \in \{1,11\} \bullet \overline{\{0,01\}^*} \bullet (\{00\} \cup \{1\}^*)$?

Challenges Involving Regular Expressions

1. Is there a simple procedure to test if a given string satisfies a regular expression?

$$11110010 \in \{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*) \qquad ???$$

2. Regular expression for all palindromes (strings which equal their reversal)?

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**
2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow \; 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}},$ **[constructor rules]**
   $\qquad\qquad\qquad\qquad\quad 1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}$.
3. Nothing else is in $\mathcal{L}_{\text{palindrome}}$. **[minimality]**

Pop Quiz. Similar looking languages: $\{0^{\bullet n}1^{\bullet k} \mid n, k \geq 0\}$ and $\{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$

Give recursive definitions of these languages.

Give regular expressions for these languages.

These computing problems look similar.

They are **VERY** different. Which do you think is more "complex"?

How to define complexity of a computing problem?

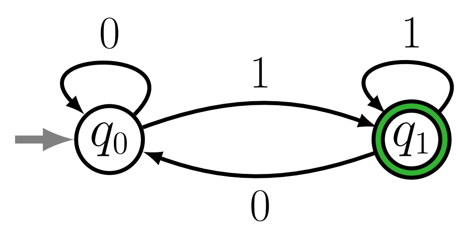
Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem ↔ "complex" YES-set ↔ hard to test membership in YES-set

How do we test membership? That brings us to *Models Of Computing*.

1 1 0 1

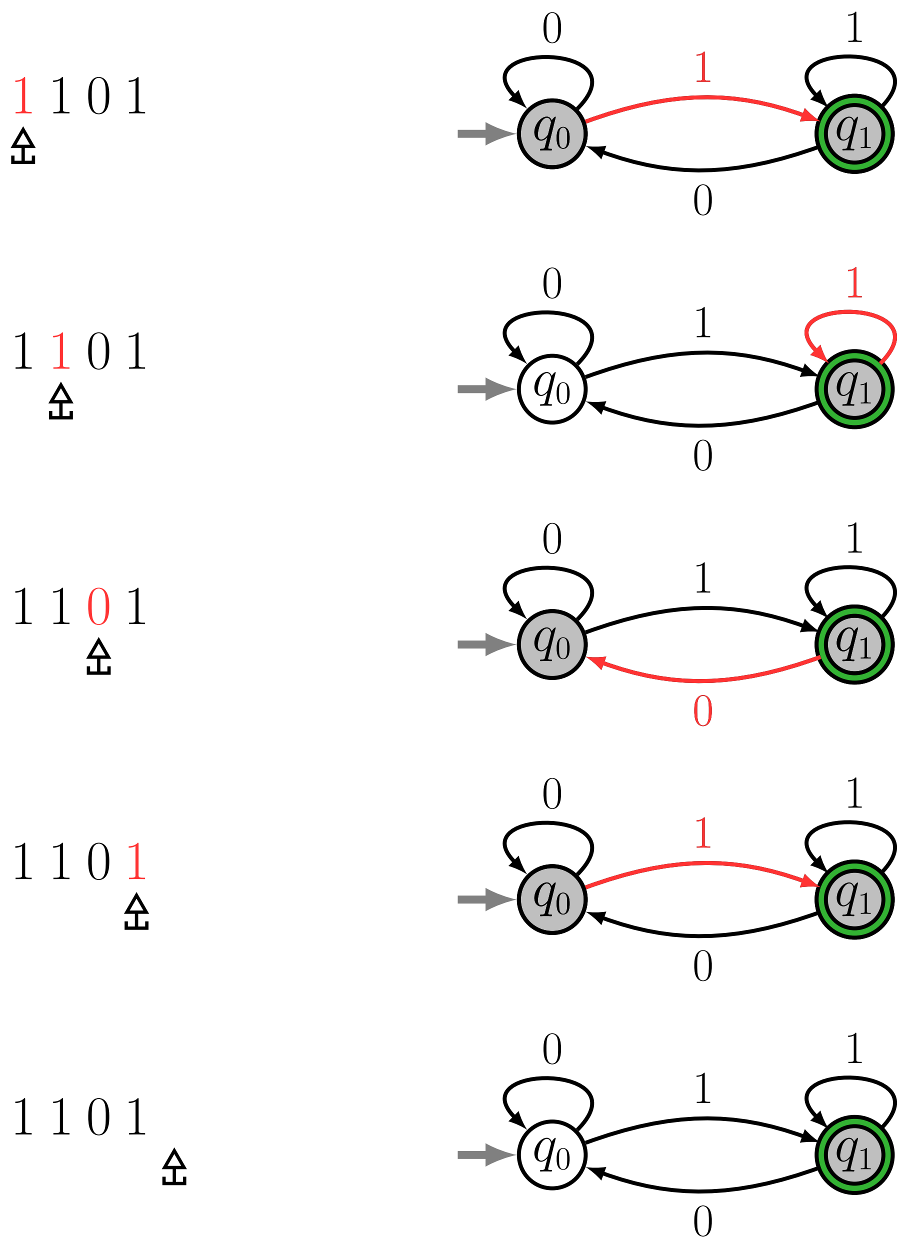


Visual encoding of four (machine-level) instructions:

1: In state q_0, when you process a 0, transition to state q_0.
2: In state q_0, when you process a 1, transition to state q_1.
3: In state q_1, when you process a 0, transition to state q_0.
4: In state q_1, when you process a 1, transition to state q_1.

"Easy" to implement as a mechanical device.

A Simple Computing Machine (DFA)



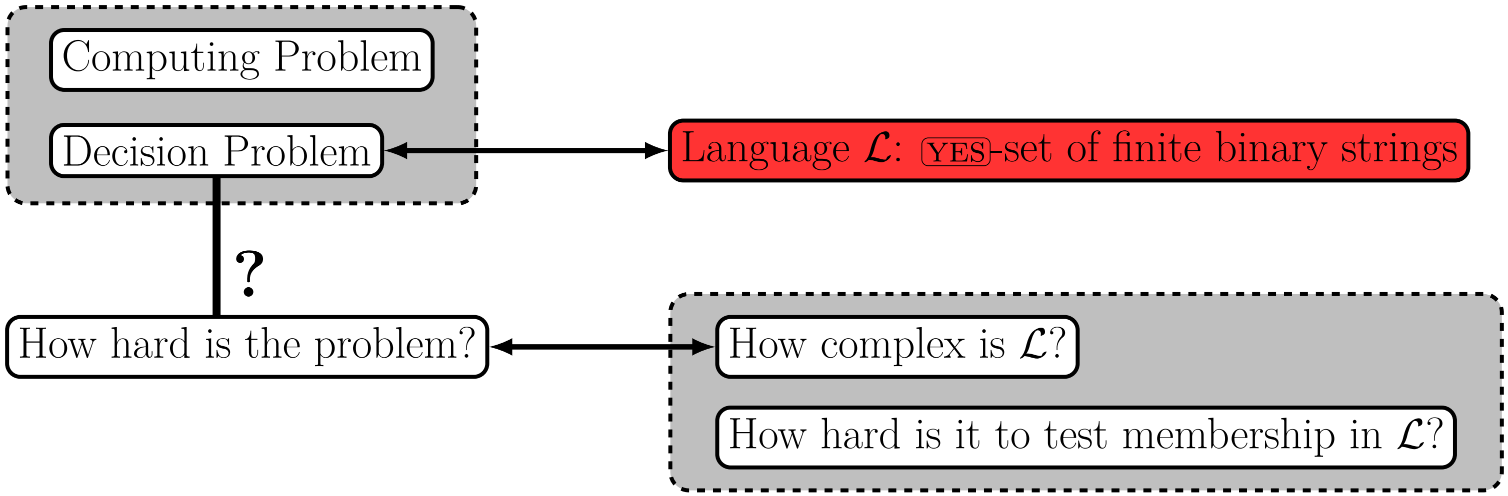
(current state in gray)

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, \ldots\}$

Strings in $\mathcal{L}_{\text{push}}$ end in the "accepting" state q_1.

Strings not in $\mathcal{L}_{\text{push}}$ do not.

Computing Problems and Their Difficulty



A problem can be harder in two ways.

1. The problem needs more resources. For example, the problem can be solved with a similar machine to ours, except with more states.
2. The problem needs a different *kind* of computing machine, with superior capabilities.

The first type of "harder" is the focus of a follow-on algorithms course.

We focus on what *can and can't be solved* on a particular kind of machine.