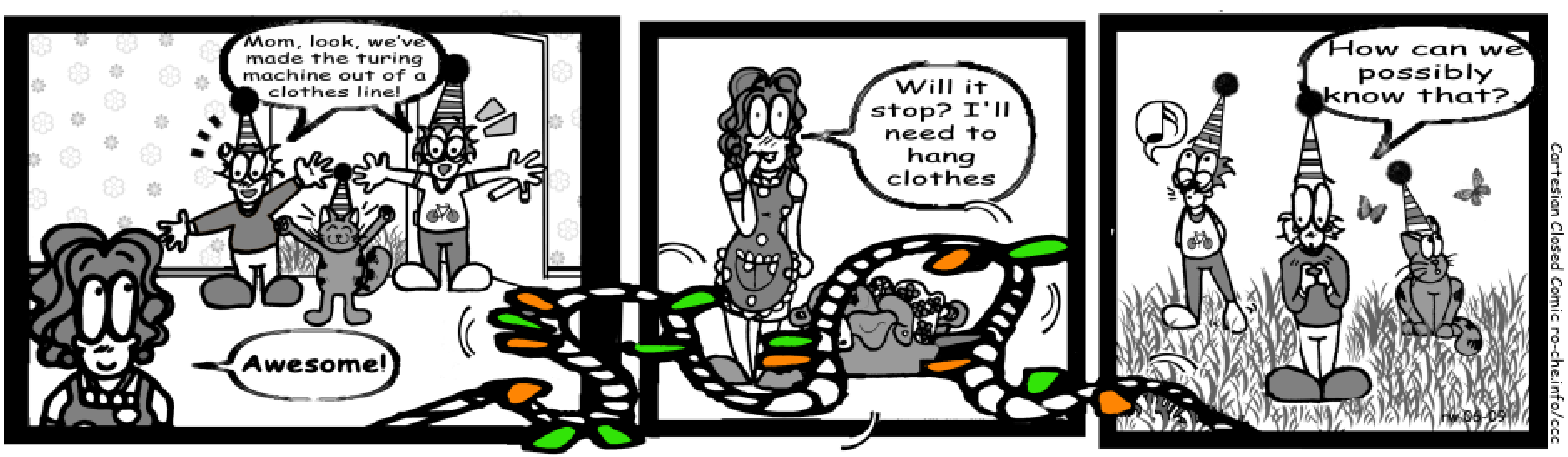
Foundations of Computer Science
Lecture 27

Unsolvable Problems

No Automatic Program Verifier for Hello-World
No Ultimate Debugger or Algorithm for PCP
The Complexity Zoo



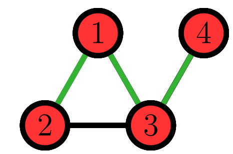
Last Time: Turing Machines

Intuitive notion of algorithm	$\equiv$	Turing Machine
Solvable problem	$\equiv$	Turing-*decidable*

$$\mathcal{L} = \{\langle G \rangle \mid G \text{ is connected}\}$$

$$\langle G \rangle = \texttt{2; 1; 3; 4 \# 1,2; 2,3; 1,3; 3,4}$$

($\langle G \rangle$ is the encoding of graph G as a string.)



M = Turing Machine that solves graph connectivity

input: $\langle G \rangle$, the encoding of a graph G.

1: Check that $\langle G \rangle$ is a valid encoding of a graph and mark the first vertex in G.

2: REPEAT: Find an edge in G between a marked and an unmarked vertex.
Mark the unmarked node or GOTO step 3 if there is no such edge.

3: REJECT if there is an unmarked vertex remaining in G; otherwise ACCEPT.

To tell your friend on the other coast about this fancy Turing Machine M, encode its description into the bit-string $\langle M \rangle$ and send over the telegraph.

You want to solve a different problem? Build another Turing Machine!

Today: Unsolvable Problems

1. Programmable Turing Machines.

2. Examples of unsolvable problems.
   - Post's Correspondence Problem (PCP)?
   - HALFSUM?
   - AUTO-GRADE?
   - ULTIMATE-DEBUGGER?

3. $\mathcal{L}_{\text{TM}}$: The language recognized by a Universal Turing Machine.
   - $\mathcal{L}_{\text{TM}}$ is undecidable – cannot be solved!

4. AUTO-GRADE and ULTIMATE-DEBUGGER do not exist.

5. What about HALFSUM?

Programmable Turing Machine: Universal Turing Machine

A Turing Machine M has a binary encoding $\langle M\rangle$. Its input w is a binary string.

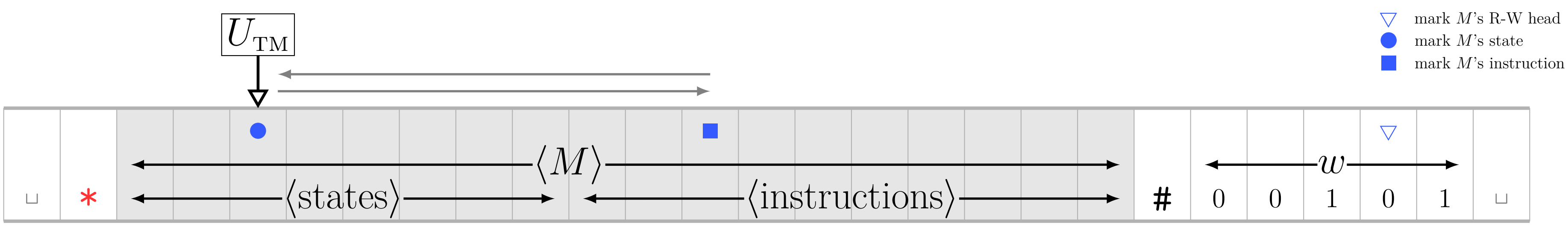
$\langle M\rangle\#w$ can be the input to another Turing Machine U_{TM}.

$$U_{\text{TM}}(\langle M\rangle\#w) = \begin{cases} \text{halt with ACCEPT} & \text{if } M(w) = \text{halt with ACCEPT;} \\ \text{halt with REJECT} & \text{if } M(w) = \text{halt with REJECT;} \\ \text{loop forever} & \text{if } M(w) = \text{loop forever;} \end{cases}$$

computer (→ U_{TM}) program (→ $\langle M\rangle$) program input (→ w)

U_{TM} outputs on $\langle M\rangle\#w$ whatever M outputs on w. U_{TM} *simulates* M

Challenge: U_{TM} is fixed but can simulate any M, even one with a million states.



Entire simulation is done on the tape.

Post's Correspondence Problem (PCP) and HalfSum

PCP: Consider 3 dominos: $d_1 = \left[\frac{0}{100}\right]$, $d_2 = \left[\frac{01}{00}\right]$, $d_3 = \left[\frac{110}{11}\right]$

$$d_3 d_2 d_3 d_1 = \left[\frac{110}{11}\right]\left[\frac{01}{00}\right]\left[\frac{110}{11}\right]\left[\frac{0}{100}\right] = \left[\frac{110011100}{110011100}\right]$$

← Top and bottom strings match. That's the goal.

INPUT: Dominos $\{d_1, d_2, \ldots, d_n\}$. For example $\left\{\left[\frac{10}{101}\right], \left[\frac{011}{11}\right], \left[\frac{101}{011}\right]\right\}$.

TASK: Can one line up finitely many dominos so that the top and bottom strings match?

HalfSum: Consider the multiset $S = \{1, 1, 1, 3, 4, 4, 5, 6, 9\}$, and subset $A = \{1, 3, 4, 9\}$.

$$\text{sum}(A) = 17 = \tfrac{1}{2} \times \text{sum}(S).$$

INPUT: Multiset $S = \{x_1, x_2, \ldots, x_n\}$. For example, $S = \{1, 1, 1, 3, 4, 4, 5, 6, 9\}$.

TASK: Is there a subset whose sum is $\frac{1}{2} \times \text{sum}(S) = \frac{1}{2} \times (x_1 + x_2 + \cdots + x_n)$?

Auto-Grade

Your first CS assignment: Write a program to print "Hello World!" and halt.

CS1: 700+ submissions!

Naturally, we do not grade these by hand.

Auto-Grade: runs each submission and determines if its correct. ←**program verification**

What does Auto-Grade say for this program:

```
n = 4;
while(n > 0){
 if(n is not a sum of two primes){
  print("Hello World!") and exit;
 }
n ← n + 2;
}
```

ULTIMATE-DEBUGGER

Wouldn't it be nice to have the ULTIMATE-DEBUGGER. ← solves the *Halting Problem*

HALTS (

```
n = 4;
while(n > 0){
  if(n is not a sum of two primes){
    print("Hello World!") and exit;
  }
n ← n + 2;
}
```

) = { YES if program halts; NO if program infinitely loops }

- We can grade the students program correctly.
- We can solve Goldbach's conjecture.
- Just think what you could do with ULTIMATE-DEBUGGER.
  - No more infinite looping programs.

Verification: Does A Program Successfully Terminate?

$$\mathcal{L}_{\text{TM}} = \{\langle M\rangle\#w \mid M \text{ is a Turing Machine and } M \text{ accepts } w\}.$$

U_{TM} is a *recognizer* for $\mathcal{L}_{\text{TM}}$.

Is there a Turing Machine A_{TM} which **decides** $\mathcal{L}_{\text{TM}}$?

- A decider must *always* halt with an answer.
- U_{TM} may loop forever if M loops forever on w.
- Question: What do these mean: $M(\langle M\rangle)$ and $A_{\text{TM}}(\langle M\rangle\#\langle M\rangle)$?

A diabolical Turing Machine D built from A_{TM}:

D = "Diagonal" Turing Machine derived from A_{TM} (the decider for $\mathcal{L}_{\text{TM}}$)
input: $\langle M\rangle$ where M is a Turing Machine.
1: Run A_{TM} with input $\langle M\rangle\#\langle M\rangle$.
2: If A_{TM} accepts then REJECT; otherwise (A_{TM} rejects) ACCEPT

D does the *opposite* of A_{TM}. Is D a decider?

Theorem. A_{TM} does not exist ($\mathcal{L}_{\text{TM}}$ Cannot be Solved)

A_{TM} exists $\rightarrow$ D exists.

D exists means it will appear on the list of all Turing Machines,

$$\langle M_1\rangle, \langle M_2\rangle, \langle M_3\rangle, \langle M_4\rangle, \langle D\rangle, \ldots$$

Consider what happens when M_i runs on $\langle M_j\rangle$, that is $A_{\text{TM}}(\langle M_i\rangle\#\langle M_j\rangle)$.

$A_{\text{TM}}(\langle M_i\rangle\#\langle M_j\rangle)$	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle D\rangle$	$\cdots$
$\langle M_1\rangle$	ACCEPT	ACCEPT	REJECT	ACCEPT	ACCEPT	$\cdots$
$\langle M_2\rangle$	REJECT	REJECT	REJECT	ACCEPT	ACCEPT	$\cdots$
$\langle M_3\rangle$	ACCEPT	ACCEPT	REJECT	REJECT	ACCEPT	$\cdots$
$\langle M_4\rangle$	ACCEPT	REJECT	REJECT	REJECT	ACCEPT	$\cdots$
$\langle D\rangle$	REJECT	ACCEPT	ACCEPT	ACCEPT	ACCEPT? REJECT?	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$D(\langle M_i\rangle)$ does the *opposite* of $A_{\text{TM}}(\langle M_i\rangle\#\langle M_i\rangle)$.

ULTIMATE-DEBUGGER and AUTO-GRADE Don't Exist

No *general* program/algorithm to analyze *any* other program M and tell if M will accept or not a particular input. ☹

→ No ULTIMATE-DEBUGGER to analyze other programs and tell if they halt. ☹

→ No AUTO-GRADE for CS-1 programs. ☹

→ No solver for PCP. ☹

Suppose ULTIMATE-DEBUGGER H_{TM} exists and *decides* if any other program halts.

We can use H_{TM} to construct a solver A_{TM} for $\mathcal{L}_{\text{TM}}$.

A_{TM} = Turing Machine derived from H_{TM} (the decider for $\mathcal{L}_{\text{HALT}}$)

input: $\langle M\rangle\#w$ where M is a Turing Machine and w an input to M.

1: Run H_{TM} on input $\langle M\rangle\#w$. If H_{TM} rejects, then REJECT.

2: Run U_{TM} on input $\langle M\rangle\#w$ and output the decision U_{TM} gives.

Exercise. Show that AUTO-GRADE does not exist.

Exercise. Show that HALFSUM is solvable by giving a decider.

The Landscape