Chapter 12

American Put Option

Recall that the American option has strike \( K \) and maturity \( T \) and gives the holder the right to exercise at any time in \([0, T]\). The American option is not straightforward to price in the Monte Carlo framework that we have discussed. The reason is that the derivative cash flow function \( f(S, t) \) is not well defined. The problem is that we cannot compute the derivative cash flow until we know how the American option is going to be exercised. If, on the other hand, we knew the exercise strategy, then it would be a straightforward task, using Monte Carlo, to obtain the expected discounted cashflows, and hence the price.

Let’s first define what an exercise strategy is. Denote an exercise strategy by \( \pi(S, t) \), which is a binary valued function of two variables, the price and the time. The exercise strategy \( \pi(S, t) \) specifies whether to exercise or not at the state \((S, t)\),

\[
\pi(S, t) = \begin{cases} 
1 & \text{exercise in state } (S, t), \\
0 & \text{do not exercise in state } (S, t).
\end{cases}
\]

12.1 Review of the Risk Neutral Stock Dynamics

Remember that all pricing occurs in the risk-neutral world, which is governed by the Martingale measure. Let’s first recall the stock dynamics in the risk neutral world,

\[
\begin{align*}
    dS &= rSdt + \sigma SdW, \\
    d\log S &= (r - \frac{1}{2} \sigma^2)dt + \sigma dW,
\end{align*}
\]

where \( dW \) is a more formal way to write \( \sqrt{dt} \epsilon(t) \) with \( \epsilon(t) \) being a zero mean unit variance independent random variable, and \( \sigma \) is the real world volatility of the stock. We have alternatively written this random process as

\[
S(t + \Delta t) = S(t)e^\eta,
\]

where \( \eta \sim N((r - \frac{1}{2} \sigma^2)\Delta t, \sigma^2 \Delta t) \). Since \( S = e^{\log S} \) and \( \log S \) is a real valued random process, this means that \( S \) is a positive valued random process. Intuitively, \( S \) has a reflective barrier at zero.
The expected move in $S$ is an increase by a factor $e^{r\Delta t}$, a consequence of the risk neutral dynamics because all prices are martingales,

$$S(t) = E[e^{-r\Delta t}S(t + \Delta t)].$$

Intuitively, as $S$ gets closer to 0, it will tend to move up by larger additive factors than down. When $S$ is far from 0, this is still the case, but the asymmetry about $Se^{r\Delta t}$ will not be as severe. This intuition has an important implication for the American call option, namely that it is never optimal to exercise early.

### 12.2 The American Call Option

If at any time the stock $S$ is below the strike $K$, then there is no reason to exercise. If, on the other hand, $S > K$, there is a choice to be made. Should one exercise now, and obtain an instant profit of $S - K$, or wait in the hopes that $S$ increases fast enough (to offset discounting). If $S$ increases fast enough, we can exercise later and make more money.

The intuition above that the stock is more likely to increase faster than $e^{r\Delta t}$ than slower seems to suggest that it is better to wait and exercise later. This argument seems applicable to any time, thus it should be *always* better to wait. This intuition seems to lead to the bizzare conclusion that it is never optimal to exercise the American call option before expiry. In this case, the American call option is exactly a European call option, and so its price is also exactly the same as that of the European call. Is there something wrong with the intuition? While it seems plausible that $S$ is more likely to increase than decrease, does it always increase at a fast enough rate? The answer is yes, and we are in fact led to the following theorem.

**Theorem 12.1.** The American call option and the European call option are equivalent.

To prove this, we will simply show that it is never optimal to exercise. Consider time $t$ where a decision to exercise appears for the first time, i.e., $S(t) - K > 0$. The cash flow from exercising is thus $S(t) - K$. Consider now the alternative strategy of waiting for a small time $\Delta t$ and then exercising. Let's compute the expected discounted cash flow for this strategy (which is available to us, since we hold an American option). We want $E[e^{-r\Delta t}(S(t + \Delta t) - K)^+]$. Since $(S(t + \Delta t) - K)^+ \geq S(t + \Delta t) - K$, we have

$$E[e^{-r\Delta t}(S(t + \Delta t) - K)^+] \geq E[e^{-r\Delta t}(S(t + \Delta t) - K)],$$

$$= E[e^{-r\Delta t}S(t + \Delta t)] - e^{-r\Delta t}K,$$

$$\stackrel{(a)}{=} S(t) - e^{-r\Delta t}K, \quad (12.1)$$

where $(a)$ follows because we are in the risk neutral world (Martingale world), which means that in this world, the price today of every instrument $(S(t))$ is the expected discounted price tomorrow $(E[e^{-r\Delta t}S(t + \Delta t)])$. Thus we see that by waiting a fixed period $\Delta t$, the expected discounted cash flow is larger in the risk neutral world. Since waiting for a fixed period $\Delta t$ is only a subset of the options available to the holder of the American option, by waiting and *optimally* exercising later, we should be able to access even higher discounted cash flows. We conclude that for every time $t$ where we have the choice to exercise, it is better to wait. This concludes the proof of Theorem 12.1. Note that the proof we have given is not specific to the GBM risk neutral process, and in fact applies to any risk neutral process.
The exercise strategy can be represented by an optimal exercise threshold function $\pi^*$ for any Markovian risk neutral dynamics. The upshot of all this discussion is that the optimal exercise strategy of waiting to exercise at time $t$ is optimal for the American call option. Consider a time $t$ when $K - S(t) > 0$, i.e. the holder has a decision to make as to exercising or not. If we go through the same analysis as above for the fixed strategy of waiting to exercise at time $\Delta t$, we obtain

$$E[e^{-r\Delta t}(K - S(t + \Delta t))^+] \geq e^{-r\Delta t}K - S(t).$$

The $\geq$ arises from the fact that $(K - S(t + \Delta t))^+ \geq K - S(t + \Delta t)$. Now the RHS is strictly less that $K - S(t)$ and all we know is that the LHS is at least the RHS so we cannot conclude anything about whether it is better to exercise now or wait a small time $\Delta t$ and exercise. In fact one might argue that if $K - S(t) > 0$ and $\Delta t$ is small enough, approaching 0, then this $\geq$ becomes close enough to equality - i.e., for small enough $\Delta t$, $e^{-r\Delta t}K - S(t) \approx E[e^{-r\Delta t}(K - S(t + \Delta t))^+] < K - S(t)$. Thus it looks like by waiting a fixed time $\Delta t$ and then exercising, one obtains smaller cash flow. This is in fact true, and so it is always better to exercise now than wait a (small) fixed time and exercise later. However, remember that waiting and exercising after a small fixed time is only a subset of the options available to the holder of the American put option – if the holder decides to wait, then he will exercise optimally later. Optimal exercise in the future has a value at least that of waiting for a fixed period and exercising, but if the additional value of optimal exercise over waiting a fixed time cannot overcome the discounting ($e^{-r\Delta t}$), then it may be optimal to exercise now than wait and optimally exercise later. Thus, we are not in the same boat as the American call option –

What can we say about the optimal exercise strategy? We can get some general properties of the optimal exercise strategy. In particular, the two properties that we would like to establish are that

1. At every time $t$, there is an optimal exercise price point $\pi^*(t)$. Below the price $\pi^*(t)$, it is optimal to exercise at time $t$ and above this price, it is optimal not to exercise, holding and optimally exercising later, over $(t, T]$. The function $\pi^*(t)$ defines an optimal exercise boundary.

2. The optimal exercise boundary $\pi^*(t)$ is non-decreasing, with $\pi^*(T) = K$.

The first property is true for certain price processes in the risk neutral world, and the second is true for any Markovian risk neutral dynamics. The upshot of all this discussion is that the optimal exercise strategy can be represented by an optimal exercise threshold function $\pi^*(t)$. The optimal exercise strategy is then given by

$$\pi^*(S,t) = \begin{cases} 1 & S < \pi^*(t), \\ 0 & S \geq \pi^*(t). \end{cases}$$

Further, the optimal exercise threshold function $\pi^*(t)$ is a monotonically increasing function of $t$. The situation is illustrated in the figure, where the shaded region indicates the states where it is not optimal to exercise.
We will now prove the second property for any Markovian risk neutral dynamics, in particular our GBM risk neutral dynamics. Suppose that it is optimal to exercise in state \((S, t)\). This means that the cash flow from exercising, equal to \(K - S\) is at least the expected discounted cash flow from optimal exercise over \((t, T]\) starting from price \(S\) at time \(t\). In particular, for any exercise strategy, \(K - S \geq E[\text{discounted cash flow for exercise according to any exercise strategy } \pi]\).

Now consider the state \((S, \tau)\) for any \(\tau > t\). Immediate exercise gives the cash flow \(K - S\), just as in the state \((S, t)\). Suppose that it is not optimal to exercise the option. In this case, there must exist some exercise strategy \(\pi\) which yields an expected discounted cash flow greater than \(K - S\), where we follow the exercise strategy \(\pi\) starting from state \((S, \tau)\). Now consider using this strategy \(\pi\) starting at time \(t\) with the stock at price \(S\), i.e., starting from the state \((S, t)\). Since the risk neutral dynamics is Markovian, the price dynamics starting from state \((S, t)\) over a time period of length \(T - \tau\) are exactly the same as the dynamics starting from state \((S, \tau)\) over the time period of length \(T - \tau\) (to maturity), because the future dynamics over a time period of length \(T - \tau\) only depends on the current price which in both cases is \(S\). Thus, starting from state \((S, t)\) and following exercise strategy \(\pi\) one has the same expected discounted cashflows over the time interval \((t, t + T - \tau]\) as one would have from following strategy \(\pi\) from state \((S, \tau)\) to the maturity of the option. By assumption, this expected cash flow is greater than \(K - S\), and so we have a strategy \(\pi\) which starting from state \((S, t)\) produces an expected discounted cash flow greater than \(K - S\), which is a contradiction. Thus, we conclude that if it is optimal to exercise in state \((S, t)\), then it is optimal to exercise in state \((S, \tau)\) for every \(\tau \geq t\). This proves that \(\pi^*(t)\) is non-decreasing in \(t\).

Proving that there is a well defined exercise point \(\pi^*(t)\) below which it is optimal to exercise and above which it is optimal to hold is a little tricky and depends on the particular risk neutral dynamics. The approach is to show that if \(\pi^*(S, t) = 1\) then \(\pi^*(S', t) = 1\) for all \(S' \leq S\). We give an intuitive argument and leave a more formal argument to an exercise. Suppose that it is optimal to exercise at \((S, t)\). Then, intuitively, it is better to take the money \(K - S\) and run, than wait and optimally exercise later. Thus, intuitively, it should be optimal to exercise at \((S', t)\) for all \(S' < S\), since one is getting more money. This becomes particularly so since due to the reflecting barrier at 0, the stock is “more likely” to move up (relative to \(e^{\sigma t}\)) than down and hence it is even more imperative to take the money and run – since the stock price is even more likely to go up from \(S'\) than it was from \(S\), one should definitely take the money \(K - S'\) if it was already optimal to take \(K - S\) and run since the asymmetry in the up versus down moves has gotten worse. The next exercise gives a slightly more formal discussion of this statement, which should probably be skipped on a first reading.

\[1\] in the sense that every price path has the same probability
12.4 Dynamic Programming on the Binomial Tree

Our first algorithm will be based on a dynamic programming approach for the binomial tree risk neutral dynamics of a stock. We briefly recap the binomial tree model for the risk neutral dynamics. The binomial tree is illustrated in the figure below for three time steps.
in which the parameters $\lambda_{\pm}$ are given by

$$
\lambda_+ = e^{(\mu - \sigma \sqrt{\Delta t}) t}, \\
\lambda_- = e^{(\mu + \sigma \sqrt{\Delta t}) t}.
$$

Here $\mu, \sigma$ are the real world drift and volatility, and $p$ is arbitrary. We have dropped the subscript $R$ for simplicity of notation. The risk neutral probability is given by

$$
\tilde{p} = \frac{e^{\mu \Delta t} - \lambda_-}{\lambda_+ - \lambda_-}.
$$

In the $n$ step binomial tree, there are $n$ discrete time steps, $\Delta t, 2\Delta t, \ldots, n\Delta t$, with $n\Delta t = T$ ($T$ is the maturity of the option). At time step $i\Delta t$, there are $i + 1$ possible stock values, $\lambda_+^k \lambda_-^{n-k} S$ for $k = 0, \ldots, i$. Thus we can use the pair of indices $(i, k)$ to index a node on the binomial tree (it is actually a grid). Note that the total number of nodes on the grid is $1 + 2 + \cdots + n + (n+1) = \Theta(n^2)$.

Let’s introduce two quantities, $V(i, k)$ and $\pi^*(i, k)$ to denote the value of holding the option and being at node $(i, k)$ and the optimal exercise strategy (1 or 0) evaluated on $(i, k)$. Remember that $(i, k)$ stands for the time $i \Delta t$ at which the stock price is $\lambda_+^k \lambda_-^{n-k} S$. Clearly $V(0, 0)$ is the price of the option. From $\pi^*(i, k)$ one can easily compute the optimal exercise threshold function $\pi^*(i)$. We will thus focus on evaluating the quantities $V(i, k)$ and $\pi^*(i, k)$ efficiently.

Consider the last time step, $n$ with the nodes $(n, k), k \in [0, n]$. Since at time $T$, the only option is to exercise if it is profitable to do so, it is clear that

$$
\pi^*(n, k) = \begin{cases} 
1 & K - \lambda_+^k \lambda_-^{n-k} S \geq 0, \\
0 & K - \lambda_+^k \lambda_-^{n-k} S < 0,
\end{cases}
$$

$$
V(n, k) = (K - \lambda_+^k \lambda_-^{n-k} S)^+.
$$

We now show how to compute $V(i - 1, k)$ and $\pi^*(i - 1, k)$ for all $k \in [0, i - 1]$ given that we know $V(i, k')$ for all $k' \in [0, i]$. Once we have done this, the algorithm will be clear. We start with $\{V(n, k)\}_{k=0}^n$ and $\{\pi^*(n, k)\}_{k=0}^n$ which are known. We then compute $\{V(n - 1, k)\}_{k=0}^{n-1}$ and $\{\pi^*(n - 1, k)\}_{k=0}^{n-1}$ from $\{V(n, k)\}_{k=0}^n$, and so on, proceeding backwards to $V(0, 0)$.

Consider $V(i - 1, k)$. In state $(i - 1, k)$ there are two options: exercise immediately if the option is in the money, in which case the cash flow is $(K - \lambda_+^k \lambda_-^{i-1-k} S)^+$; or, wait. If we wait, there are...
two possible scenarios: the stock goes up to \( \lambda^k_{i+1} \lambda^{-1-k} S \), i.e. to the node \((i, k+1)\); or, the stock goes down to \( \lambda^k_{i} \lambda^{-k} S \), i.e. to the node \((i, k)\). The values \( V(i, k+1), V(i, k) \) of being in both of these states are known (by assumption). Thus, the expected discounted value (under the risk neutral measure) of holding can be computed as

\[
V_h(i - 1, k) = e^{-r \Delta t} (\tilde{p} V(i, k + 1) + (1 - \tilde{p}) V(i, k)).
\]

Similarly, the value of exercising is

\[
V_{ex}(i - 1, k) = (K - \lambda^k_{i} \lambda^{-1-k} S)^+.
\]

If the cash flow of exercising is at least the expected cash flow of waiting, then it is optimal to exercise, and vice versa. Thus, we conclude that

\[
\pi^*(i - 1, k) = \begin{cases} 
1 & V_{ex}(i - 1, k) \geq V_h(i - 1, k), \\
0 & V_{ex}(i - 1, k) < V_h(i - 1, k),
\end{cases}
\]

\[
V(i - 1, k) = \max\{V_{ex}(i - 1, k), V_h(i - 1, k)\}.
\]  

Equations (12.5) and (12.6) are the key steps in the dynamic programming algorithm. Notice that every node of the tree will be visited, and at each node a constant amount of work is done, so the running time of the algorithm is \( \Theta(n^2) \). The quadratic running time is essentially unavoidable. If all the values \( \pi^*(i, k) \) and \( V(i, k) \) are stored, then the memory requirement is also \( \Theta(n^2) \), which for \( n \) reasonably large is unmanageable. Luckily, it is possible to run the algorithm with only a memory requirement \( \Theta(n) \), to store the exercise function \( \pi^*(i) \) and obtain the price \( V(0, 0) \). The main idea is that once \( V(i - 1, k) \) is computed, \( V(i, k') \) will never be needed again, and so that memory can be reused.

**Exercise 12.3**

Show more explicitly that only \( \Theta(n) \) memory is needed for the dynamic programming algorithm. Try to minimize the memory requirement as much as you can.

We give the full algorithm in pseudo code below.

1. **Algorithm:** Pricing American Put using Dynamic Programming
2. Select \( \Delta t \) and compute \( \lambda_{\pm} \) and \( \tilde{p} \).
3. Initialize \( \pi^*(n) = K \) and a vector \( v \) of size \( n + 1 \) to \( v_k = V(n, k) \) for \( k \in [0, n] \).
4. **for** \( i = n - 1 \) to 0 **do**
5. **for** \( k = 0 \) to \( i \) **do**
6. \( v_k \leftarrow \max\{(K - \lambda^k_{i} \lambda^{-k} S)^+, e^{-r \Delta t} (\tilde{p} v_{k+1} + (1 - \tilde{p}) v_k)\} \)
7. **if** \( v_k \leq (K - \lambda^k_{i} \lambda^{-k} S)^+ \) **then**
8. \( \pi^*(i) = \lambda^k_{i} \lambda^{-k} S \)
9. \( v_0 \) is the option price.
10. \( \pi^*(i) \) contains the optimal exercise threshold function.

The algorithm as stated above is perfectly fine for an infinite precision machine, however on a finite precision machine, a certain amount of care needs to be taken. In particular, lets consider the “simple” task of initializing \( v \) in step 3 of the algorithm. This involves computing the stock price \( s_k \) since \( v_k = (K - s_k)^+ \), so we consider the initialization of \( s_k \), for \( k = 0, \ldots, n \). A natural approach would be to first initialize \( s_0 = \lambda^o S \) and then initialize the remaining \( s_k \) for \( k \in [1, n] \).
using the update $s_k = \frac{\lambda_+}{\lambda_-} s_{k-1}$. Let’s consider the numerical value of $s_0 = \lambda^n S$. Note that for $\Delta t = \frac{T}{n}$ small enough, $\sqrt{\Delta t} \gg \Delta t$ and so $\lambda_- \approx e^{-\sigma \sqrt{\frac{T}{n}} \sqrt{p}}$. Thus,

$$\lambda^n \approx e^{-\sigma \sqrt{nT} \sqrt{\frac{p}{1-p}}}.$$ 

Thus, for large $n$, to within the numerical precision of most computers, $s_0$ will evaluate to 0, and if the update $s_k = \frac{\lambda_+}{\lambda_-} v_{k-1}$ is used with starting condition $s_0 = 0$, all the $s_k$ will be zero, and the algorithm will be doomed from the beginning.

**Exercise 12.4**

Give a better approach to initializing the vector $v$ than the one discussed above.

[Hint: Consider computing $\log s_k$. Show that $\log s_k = k \log \lambda_+ + (n-k) \log \lambda_- + \log S$.]

As an example, running our dynamic programming algorithm with $S = K = 100$, $T = 2$ years, $r = 0.05$ (annualized), $\sigma = 0.2$ (annualized) and $n = 200,000$, the American put option price was 7.723197. The optimal exercise threshold function is given in the figure below.

![Optimal Exercise Threshold Function](image)

As one increases $n$, the price converges to the true price, hence this method is one of a class of methods known as convergent methods. However, the behavior of this convergence is interesting. It is not monotonic, so the price cannot be used to bound the true price in any systematic way. It is known that for the Binomial tree with $n$ discretization time steps, the absolute error in the price converges at a rate $\Theta\left(\frac{1}{n}\right)$. Some of these behaviors are illustrated in the figures below.
12.5  Pricing By Optimizing Bounds

12.6  Optimal Exercise from Pricing

12.7  Dividend Paying Stocks and Put-Call Parity
12.8 Problems
Chapter 13

Stochastic Differential Equations

Exercise 13.1

Assume the initial stock price is $S_0$ and it follows real and risk neutral dynamics given by

\[
\Delta S = \mu S \Delta t + \sigma S \Delta W \\
\Delta \tilde{S} = r \tilde{S} \Delta t + \sigma \tilde{S} \Delta \tilde{W}.
\]

Write a program that takes as input $\mu$, $r$, $\sigma$, $S_0$, $T$, $\Delta T$ and simulates the stock price from time 0 to $T$ in time steps of $\Delta t$ for the risk neutral world, using each of the following modes:

(a) Binomial mode I: compute $\lambda_\pm$ from $\mu$, $\sigma$ assuming that $p = \frac{1}{2}$, and then computing $\tilde{p}$.

(b) Binomial mode II: compute $\lambda_\pm$ from $\mu$, $\sigma$ assuming that $p = \frac{2}{3}$, and then computing $\tilde{p}$.

(c) Continuous mode: using the continuous risk neutral dynamics $r$, $\sigma$ generate at time step $\Delta t$ as if the discrete model were taken to the limit $dt \rightarrow 0$.

For each of the three methods, give plots of representative price paths for $S_0 = 1$, $\mu = 0.07$, $r = 0.03$, $\sigma = 0.2$, $T = 2$ using $\Delta t = 0.1, 0.01, 0.0001$. 
