Computational Finance – Options

1 Options

An option gives the holder of the option the right, but not the obligation to do something. Conversely, if you sell an option, you may be obliged to do something. An option is typically defined with respect to some underlying instrument, for example a bond or a stock. We will only consider options on stocks for the moment. All the concepts that we discuss can be illustrated with the simplest options (the plain vanilla European Call/Put options, and their American cousins). Just for informative purposes, however, we will briefly introduce Asian, Lookback and Barrier options.

1.1 The Call Option on a Stock

A call option with *strike price* K gives the holder the right to buy the stock at the price K. Suppose that the market price of the stock is S. Then if $S \geq K$, a holder of the option could buy the stock at K and then sell it at S on the market, making a profit of $S - K$. Note that whoever sold you the option is *obliged* to sell you the stock at K. Thus, if $S \geq K$, then the value of the option is $S - K$. If, on the other hand, $S \leq K$, since you hold an option, you would choose not to exercise it, for why would you buy the stock for K when you can obtain it for cheaper at S on the market. Thus, when $S \leq K$, the value of the option to you is 0. Let C be the value of the option, then we can summarize this discussion in the formula

$$C = \max(S - K, 0).$$

We will often write this expression as $(S - K)^+$.

Exercise 1.1

Plot the value of the call option C as a function of the stock price S, for a given strike K.

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We now introduce a slight twist by bringing time into the picture. Suppose instead of a call option today, we purchase a call option which gives the right to buy the stock at the strike K *at some future time* T. T is called the time to expiry, and this option is called a

Plain Vanilla European Call Option with strike K and expiry T.

If S is the current stock price, then some terminology used is that if $S > K$, this is an "in the money" option, if $S = K$ it is an "at the money" option and if $S < K$ it is an "out of the money" option. An interesting question is how much one should be willing to pay for such an option. For example, consider the following scenario:

> IBM stock is currently trading at \$100. How much would you pay for an option with strike $K = \$95$ and expiry $T = 1$ year?

Clearly, the amount you will pay should depend on your estimate of what the possible (or probable) outcomes of the stock price are a year from now. For example if you were 100% sure that the stock would drop to below \$90 in a year, then this option is worthless to you.

A further level of complexity is introduced if the call option gives you the right to buy the stock at the strike at any time between now and expiry. This option is called a

Plain Vanilla American Call Option with strike K and expiry T.

Since you can always choose to exercise the American call at time T, its value cannot be less than the value of the corresponding European call. However, the fact that you now have more flexibility to exercise at any time between 0 and T introduces a whole new level of complexity to pricing the option. Further, another issue arises, namely how to optimally exercise the option. For example, suppose that you buy an American call with strike \$100 and expiry $T = 1$ year. After $\frac{1}{2}$ a year, the stock price is \$102. Should you exercise now, or wait in the hopes that the stock price will rise even further? This type of problem in the control theory community is one of stochastic optimal stopping, because we want to know when to stop, i.e. get out of, the option when the stock price movements are stochastic.

Exercise 1.2

Show using an arbitrage argument that the American call option cannot have a lower price than the European call option.

1.2 The Put Option on a Stock

The put option is similar to the call option, except that instead of giving the holder the right to buy, it gives the holder the option to sell.

Thus, the *Plain Vanilla European Put Option with strike K and expiry T* gives the holder the option at time T to sell the stock at the strike price K. Similarily, the *Plain Vanilla American Put Option with strike K and expiry T* gives the holder the option at any time in the interval $[0, T]$ to sell the stock at the strike price K.

Exercise 1.3

In a time of a market crash, which option is more valuable to hold, a put option or a call option?

In a time of a market boom, which option is more valuable to hold, the put option or a call option?

1.3 Other Options

The Asian Option In an Asian option, the strike price, and hence the payoff, is defined in relation to the average stock price over some previous period. For example, an Asian call option contract could look something like:

> Call option with strike K equal to the average daily price over the 6 months prior to the maturity T.

Asian put options could also be similarily defined. Further, they may come in either the European or American flavor.

The Lookback Option In a lookback option, the strike price is typically defined as the minimum or maximum over some period in the past. Interestingly, many companies choose to compensate their employees in part by giving them lookback options. A typical lookback call option contract might look something like:

> Lookback call on stock S with strike being the minimum daily close price over the six months prior to the maturity T.

Such lookback options may come either in the European or American flavors.

The Barrier Option A barrier option is an option that gets activated when the price of the stock hits some barrier. The strike price could be defined as value of the barrier itself, or some other value. An example of a barrier option is given in the contract below.

> Put option with strike \$90 with activation barrier \$80 and maturity T. The option can be exercised as a normal put at time T only if the price drops below the barrier in the six months prior to the maturity.

If the current price is \$90, the barrier-put becomes valuable only when the price hits the \$80 barrier before T. On the other hand, the plain vanilla put will become valuable as long as the stock price is below \$90 at maturity T, no matter whether the barrier was hit. The barrier could also come in an American or European flavor. The American flavor will only give the right to exercise after the barrier gets hit, but before the expiry.

Mixed Options. It is possible to use mixtures of options to construct even more exotic contracts. For example a barrier option where the strike is based on some average stock price would be a mixed Asian-barrier option. In general, Asian style options are very hard to price analytically, however they can be priced using simulation. In general, the more exotic an option gets, the harder it is to price.

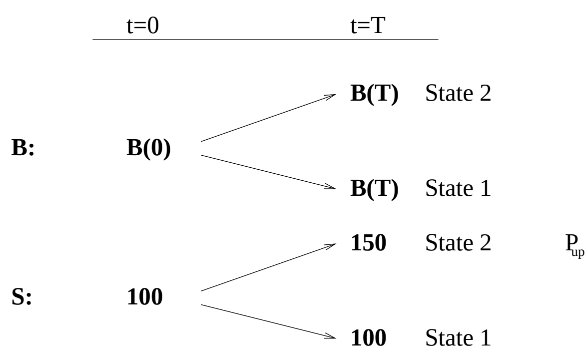
Market Crash Options. A market crash option could be an option which gets activated in a market crash. One way to define a market crash is using a barrier. An alternative is to define it with respect to the maximum Drawdown of the stock, which corresponds to a moving barrier whose value depends on the previous maximum of the stock price.

Such options can be used to insure against a market crash relative to its current value, as opposed to a barrier option which can be used to insure against a market crash to some absolute value.

2 Pricing Options

The general principle we will use to price options is that of no arbitrage. To illustrate the concept, consider the following very simple *two period economy.*

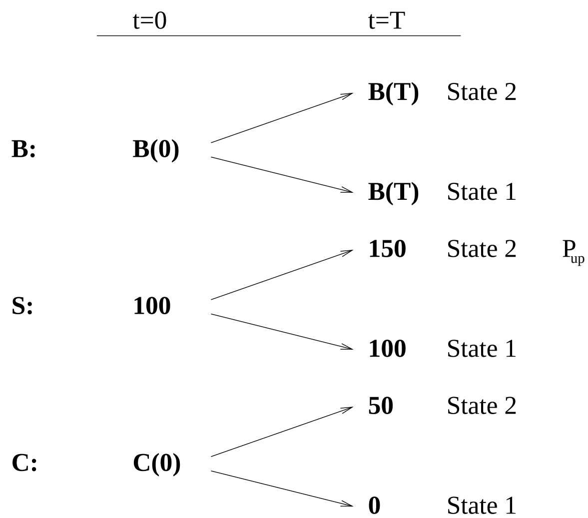
At $t = 0$, two instruments are available. Any amount (even fractional) of either instrument may be sold or puschased at the specified market price - i.e., arbitrary short or long positions are allowed. A *risk free* asset or *bond*, B, and a *stock*, S. At $t = 0$ (the first period), the bond is worth $B(0)$, and, the stock is worth $S(0) = 100$. At $t = T$ (the second period), the economy can be in one of two states. In both states, the bond is valued at *B(T)* and hence is risk free. In the first state the stock is valued at $S(T) = 100$ and in the second state, the stock is valued at $S(T) = 150$.



To complete our model for the market, we need to specify the probabilities that the market will be in states 1 and 2 respectively. Suppose that P_{up} is the probability that the market goes up. We have thus completely specified the market dynamics for our simple economy. In this simplified economy, it is clear that one can guarantee an amount $B(T)$ at $t = T$ by investing $B(0)$ in the bond at $t = 0$. The *risk free discount factor* is defined by

$$\frac{1}{D(T)} = \frac{B(0)}{B(T)} = e^{-rT} \tag{1}$$

where $r > 0$ is the risk free interest rate for continuous compounding. The economy becomes more interesting when we introduce a third instrument, a European call option with strike 100 expiring at time T, whose values at $t = T$ in the possible states of the economy are known. At $t = T$, the call option will either be worth 50, if the market went up, or 0 if not. The complete picture of the market is now summarized in the following figure



The question we would like to answer is "What should the price of the call option be at $t = 0$, i.e., what is $C(0)$?". Suppose that P_{up} is very close to 1. Holding the option will "almost" guarantee 50 at $t = T$. hence one might expect that[1] $C(0) \to 50/D(T)$ as $P_{up} \to 1$. On the other hand, if P_{up} is very close to 0, then $C(T) = 0$ with very high probability and we should not be willing to pay anything for it at $t = 0$. Hence, one might expect $C(0) \to 0$ as $P_{up} \to 0$. In general, we might expect that $C(0)$ will be some function of P_{up}. A good guess would be that the value at $t = 0$ should be the expectation of the value of the instrument at $t = T$, discounted to $t = 0$ by multiplying by $1/D(T)$. In doing this we are treating the call option as if it were a riskless asset, yielding its expected value at $t = T$. Since the option is actually risky, in general we might expect that we should not have to pay as much (since we are absorbing this risk), hence, we generally expect that

$$C(0) \leq \frac{1}{D(T)} \times E[C(T)] \tag{2}$$

The difference between the left hand side and the right hand side is sometimes denoted the *risk premium*. The surprising thing, as we are about to see, is that $C(0)$, the price to be paid for the option, is independent of P_{up}, provided that $0 < P_{up} < 1$. All the (apparently) intuitive arguments made in the last paragraph (taking P_{up} close to 1 or 0) are actually flawed. In fact, even (2) need not necessarily hold.

[1] we multiply by $1/D(T)$ to be consistent with the bond dynamics, namely, a sure $B(T)$ at time $t = T$ is worth $B(0)$ at $t = 0$, hence, a sure 50 at time $t = T$ should be worth $B(0) \times 50/B(T)$ at $t = 0$.

To understand why this is so, we need to use the no arbitrage condition. remembering that a call option is valuable when the stock goes up, a portfolio constructed by buying the stock and selling the option should have a value which is immune to the stock price. Indeed, the price of this portfolio at $t = 0$ is $100 - C(0)$. Its value in the future, at time T is 100 whether the stock goes up or down. Therefore by spending $100 - C(0)$, I can guarantee 100 at time T. It should therefore be that $B(T)/B(0) = 100/(100 - C(0))$, or that

$$C(0) = 100(1 - \tfrac{B(0)}{B(T)}) = 100(1 - \tfrac{1}{D(T)}).$$

Note that nowhere in this argument have we made any reference to P_{up}.

Informally, we then have the following *mini*–theorem. We give a proof of this by showing how to construct an arbitrage opportunity. In fact, we will even show that the discount factor, cannot take on arbitrary values.

Theorem 2.1 *If $0 < P_{up} < 1$, and the no-arbitrage axiom is valid, then*

(i) $\frac{2}{3} < \frac{1}{D(T)} < 1$;

(ii) $C(0) = 100(1 - \frac{1}{D(T)})$;

Note that these claims are independent of P_{up}!

To prove this theorem, we will actually make use of a slightly stronger defiinition of no-arbitrage than the one we have used up to now. This will become clear during the course of the argument.

Let us consider the first claim, and suppose that $\frac{1}{D(T)} \le \frac{2}{3}$. Then, at $t = 0$, sell one unit of stock and buy 100 dollars worth of bond. The net investment is zero[2]. At $t = T$ the bond is worth at least 150 (as $1/D(T) \le 2/3$) and since you owe one unit of stock, the profit in state 1 is at least 50 and in state 2 is at least 0. Thus, with no investment, you can guarantee a non-negative profit in all states, and a positive profit in some states. This is clearly a very desirable opportunity, and any rational individual would wish to consume an infinite amount of this portfolio. It is not quite a strict arbitrage, as there is no guaranteed stream of non-negative cashflows, at least one of which is positive, but it certainly smells like an arbitrage opportunity of some sorts. In fact it is an example of a more general kind of arbitrage opportunity which we will discuss later. Certainly, no economy in equilibrium can sustain such an opportunity, hence it should be disallowed. Thus, we conclude that $1/D(T) > 2/3$. Similarly, one shows that $1/D(T) < 1$ by buying one unit of stock and selling 100 dollars worth of bond.

The more interesting claim is the second claim. Apparently the price $C(0)$ should be independent of P_{up}. Consider the following portfolio: Sell one stock, buy one call option. The cash inflow is $100 - C(0)$. We can thus buy $(100 - C(0))/B(0)$ bonds. Thus we consider the portfolio

$$\Pi = \{-1 \text{ Stock}, +1 \text{ Call Option}, +\tfrac{100 - C(0)}{B(0)} \text{ Bonds}\}.$$

[2] We ignore transaction costs here and throughout.

By construction, the value of this portfolio is 0 at time 0. Let's consider its value at time T. If the stock goes up, the value Π is

$$-150 + 50 + (100 - C(0))\frac{B(T)}{B(0)} = -100 + (100 - C(0))\frac{B(T)}{B(0)}.$$

If, on the other hand, the stock goes down, the value of Π is

$$-100 + 0 + (100 - C(0))\frac{B(T)}{B(0)} = -100 + (100 - C(0))\frac{B(T)}{B(0)}.$$

Thus in each state at time T, the portfolio Π has the *same* value. If this value is greater than zero, then we have arbitrage. If on the other hand, this value is less than zero, we have an arbitrage by selling Π. Thus, we conclude that the value at time T must be equal to 0, i.e.,

$$0 = -100 + (100 - C(0))\frac{B(T)}{B(0)}.$$

The second claim now follows after solving for $C(0)$. The important things to note are:

1. The exact nature of the instruments that constitute the market are not relevant, only that they follow the specified market dynamics.

2. To price the third instrument (given $D(T)$, or alternatively, the interest rate), it was not necessary to know P_{up} (the probabilities of of the various states). One only needs to know what states are *possible*. Among other things, this implies the counter intuitive fact that the price of the option would be the same whether $P_{up} = 1 - 10^{-100}$ or $P_{up} = 10^{-100}$.

That the actual probability P_{up} plays no role in the pricing of the instruments is not a quirk of the particular economy that we happen to have chosen. We will see later, when we discuss Martingale measures, that this little result is a special case of a much more powerful theorem. Our approach here was to somehow construct a portfolio that was an arbitrage portfolio, in the sense that its value must be zero. If not, then there was arbitrage. While it is not clear how we came up with this portfolio, once we had the portfolio, the remainder of the argument seemed somewhat straightforward. Knowing what portfolio to construct may seem a little like magic at this point, but we will soon see a purely mechanical and systematic approach to getting $C(0)$.

The fundamental take home message is that

> *To price options, the guiding hand is that whatever the price be, there should be no arbitrage. Sometimes, as in our example, this alone suffices to determine the price.*