CSCI 4965/6961 | RPI
Introduction to Computational Finance | Fall 2005

FINAL

Last Name: **Student ID#:**

Instructions

Fill in your Last Name and ID#. Answer **all** *questions in the space provided. Keep your answers BRIEF! You have 1 Hour and 20 min. You may consult a double sided sheet of notes during the exam.*

1	2	3	4	5	6	7	**TOTAL**
100	**50**	**100**	**50**	**100**	**100**	**400**	**500**

1. Let $B(T)$ be the zero coupon bond price with expiry T.

   (a) **[50]**What should the price of the following contract be:

   At $t_1 > 0$ I give you \$5 and at $t_2 > t_1$ you give me \$ 4.

   Your answer should be in terms of $B(t_1)$ and $B(t_2)$.

   (b) **[50]**An instrument has cash flow *rate* $c(t), 0 \leq t \leq T$, i.e., the cash flow in the interval $[t, t + dt]$ is $c(t)dt$. Show that the price P of this instrument should be

   $$P = \int_0^T dt\ B(t)c(t).$$

   Prove your answer using arbitrage arguments. (Assume that you can buy arbitrary amounts of bond at arbitrary expirations.)

2. **[50]**Let $r(t)$ be the cumulative return to time t of a portfolio. Assume that the possible values of t are $0, 1, \ldots T$. Give a *linear* time algorithm to compute the Maximum Drawdown for this return series.

3. **[100]**Consider the single period stock - bond economy, in which at time Δt the stock couold have values $\lambda_+ S$ or $\lambda_- S$, where S is the initial price, and $\lambda_+ > \lambda_-$. Suppose that the interest rate is r. Let $\tilde{p}$ be the *risk neutral* probability to go up, i.e., to $\lambda_+ S$. Show that

$$\tilde{p} = \frac{e^{r\Delta t} - \lambda_-}{\lambda_+ - \lambda_-}.$$

What constraint does this place on $e^{r\Delta t}$, if there is to be no arbitrage? Explain the intuition.

4. **[50]**We would like to obtain an expected value by Monte Carlo. We draw 100 samples and the variance of this sample is $\sigma^2 = 9$ and the mean is 90. *Estimate* the number of samples that need to be drawn so that the *relative error* is very likely not more than 10^{-3}. The relative error is the ratio of the error to the actual value.

How would your answer change if the mean had been 0.9.

5. **[100]**A stock has the dynamics: $dS = \frac{1}{4}\sigma^2 dt + \sigma\sqrt{S}\, dW$. Show that $U = \sqrt{S}$ has the dynamics

$$dU = \frac{\sigma}{2} dW.$$

Use this to show that $S(T) = U(T)^2$, where U has a Normal distribution. What are the mean and variance of $U(T)$. The distribution of S is called a χ^2 distribution.

6. **[100]**Consider a two period problem. The rate of return r is defined as $W(T) = W(0)(1+r)$, where W represents wealth. Two stocks S_1, S_2 have Normal rate of return with mean r_1, r_2 and variance σ_1^2, σ_2^2. Let the correlation between the returns be ρ. Consider a portfolio in which we spend fraction f_1 of the initial wealth on S_1 and f_2 on S_2 ($f_1 + f2 = 1$). Give an expression for the rate of return of this portfolio. What is its distribution?

Now suppose that $r_1 = 0.05, r_2 = 0.15$, $\sigma_1 = \sigma_2 = 1$, $\rho = 0.5$ and $f_1 = f_2 = 0.5$. Give an expression for the VaR with confidence threshold $\alpha = 0.95$ and initial wealth $W(0) = 100$. Your expression can be in terms of the CDF of the standard Normal, $\phi()$, or its inverse.

7. [**400**]This problem deals with generating paths backward rather than forward. For simplicity lets consider the binomial model with probability p to go up (when generating forward).

   Suppose that at time step $t = n$, we "start" at position s. Remember that $s \in \{-n, -n+2, \ldots, n-2, n\}$. We would like to generate a path backwards, toward $t = 0$. Suppose that in general we have generated backwards to time t and are at position i. To generate the next step backward, let $P_+(t, i)$ be the probability that you move up (backwards) to end at $(t-1, i+1)$ and $P_-(t, i)$ be the probability to move down to end at $(t-1, i-1)$. Note $P_-(t, i) = 1 - P_+(t, i)$. Show that

$$P_+(t, i) = \frac{t-i}{2t}.$$

   This allows us to generate the path backward. [Hint: You may need Bayes theorem.]

   Note the remarkable thing that $P_+(t, i)$ is independent of p. But surely the paths we generate must depend on p, whether we do it forward or backward. Where is the p dependence?

SCRATCHWORK: