ASSIGNMENT 8

Homeworks are due at the begining of class on the due date. The point value for the 6000 level is indicated in small font.

1 (80 (60) points) Portfolio Optimization and Risk

Consider n instruments $S_1, \ldots, S_n$ and the two period economy at times $0, T$. Assume that the assets are all log-normal, i.e., $S_i(T) = e^{U_i}$, where $U_i \sim N(\mu_i, \sigma_i^2)$ (for simplicity, we have assumed that $S_i(0) = 1$ for all i). Assume that the correlation matrix $[\rho_{ij}]$ is given by

$$\boldsymbol{\rho} = (1-\rho)\boldsymbol{I}_n + \rho\mathbf{1}\mathbf{1}^T.$$

The matrix $\boldsymbol{\rho}$ has ones along the diagonal and ρ in every off diagonal element. Note that $Cov(U_i, U_j) = \sigma_i\sigma_j\rho_{ij}$.

Assume that $\sigma_i = 1$, $\rho = 0.05$ and that $n = 100$. Generate each μ_i's independently from a Normal distribution with mean 0.25 and variance 1.

(a) Compute the mean variance efficiency frontier. Plot it as a function of variance of wealth versus expected wealth (at time T).

(b) Obtain the portfolio which maximizes the Sharpe ratio: $\frac{\text{Exp. Wealth}}{\text{Std. Dev. of Wealth}}$, and report the optimal Sharpe, and expected wealth that attains this Sharpe.

(c) Obtain the value at risk, VaR, for the portfolio of the previous part using the normal approximation and $\alpha = 0.05$.

(d) Use a Monte Carlo to estimate the answer to the previous part and compare.

(e) Repeat (c) and (d) for $\rho = 0$. Compare and explain.

2 (20 (40) points) Portfolio Optimization

In this problem, the goal is to rephrase a portfolio optimization problem as a mathematical programming problem, in particular a mixed integer linear program (MILP).

Suppose that there are instruments $i = 1, \dots n$ with returns R_i. View the instruments as investments or activities. Investing in an instrument has some fixed cost, f_i and some per unit cost p_i. Thus if an amount $x_i > 0$ of instrument i is purchased, then the cost of this investment is $f_i + x_i p_i$. Assume further that there are minimum and maximum investment constraints if one chooses to invest in instrument i: $m_i \leq x_i \leq M_i$. So, $x_i \in \{0\} \cup [m_i, M_i]$. Assume that initial capital is W_0, and the goal is to maximize the expected return, $\sum_i x_i R_i$. Formulate this portfolio allocation problem as a mixed integer linear program (MILP).

You do not need to solve it, just formulate it.

[Hint: For each i, introduce auxiliary binary variables $y_i \in \{0, 1\}$.]

A linear program has the following general format:

$$\max_{\mathbf{x}} \boldsymbol{\theta}^T \mathbf{x}$$

subject to any number of linear inequality or equality constraints of the form:

$$\begin{aligned} \mathbf{Ax} &= \mathbf{a} \\ \mathbf{Bx} &\geq \mathbf{b} \end{aligned}$$

A MILP is similar, except that some of the variables are constrained to be in integer sets, eg $\{0, 1\}$.