FINAL: 180 Minutes

Last Name: Solutions

First Name: ____________

RIN: ____________

Section: ____________

Answer ALL questions. You may use two single sided $8\frac{1}{2} \times 11$ crib sheets.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
100	50	50	50	50	50	350

1 Circle at most one answer per question. 10 points for each correct answer and -5 points for each incorrect answer (blank answer is 0 points). Don't guess!

(a) $P(n)$ is a predicate ($n \in \mathbb{N}$). $P(1), P(2), P(3)$ are true, and $P(n) \to P(n+4)$ is true for $n \geq 1$. For which n can we be <u>sure</u> $P(n)$ is true?

$\boxed{P(1)}$ $\boxed{P(2)}$ $\boxed{P(3)}$ $\boxed{P(5)}$ $\boxed{P(6)}$ $\boxed{P(7)}$ . . .

- [A] All $n \geq 1$ except multiples of 2.
- [B] All $n \geq 1$ except multiples of 4.
- [C] All $n \geq 1$
- [D] Only $n = 1, 2, 3$.

B

(b) Of the following five sets, list *all* that are <u>countable</u> (A is countable if $\mathbb{N} \xrightarrow{\text{surj}} A$):
(I) Prime numbers; (II) Rational numbers; (III) Integers; (IV) Even numbers; (V) Infinite binary strings.

I, II, III, IV,

- [A] I and III.
- [B] I and II and III and IV.
- [C] I and III and V.
- [D] II and III and IV.

B

(c) A class with 25 students needs to choose a representative committee which is a <u>subset</u> of 5 students. How many different committees can be formed?

$\binom{25}{5} = \frac{25!}{5!\,20!}$

- [A] 25^5.
- [B] $\frac{25!}{20! \times 5!}$.
- [C] $\frac{25!}{5!}$.
- [D] $25 \times 24 \times 23 \times 22 \times 21 = \frac{25!}{20!}$.

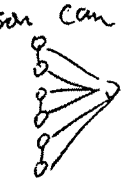
B

(d) A friendship network has 7 people and each person has at least 1 friend. 6 of the people have *exactly two friends*. How many friends can the 7th person have? <u>Give all possibilities.</u>

7th person must have even friends → 2, 4, 6.

o o o o o o o

the 7th person can have 6 friends:



- [A] The seventh person could have either 2 or 4 friends.
- [B] The seventh person could have either 2 or 4 or 6 friends.
- ~~[C]~~ The seventh person could have either 1 or 2 or 3 friends.
- ~~[D]~~ The seventh person could have any number of friends that is greater than 1.

B

(e) Compute the summation $(0+1) + (1+2) + (2+4) + (3+8) + \cdots + (10 + 2^{10}) = \sum_{i=0}^{10}(i + 2^i)$

$$\sum_{i=0}^{10}(i+2^i) = \sum i + \sum 2^i = \frac{10 \cdot 11}{2} + 2^{11} - 1$$
$$= 55 + 2047$$
$$= 2102.$$

- [A] 2048.
- [B] 2102.
- [C] 1078.
- [D] 2200.

B

(f) You have a known fact that $0 = 0$ and all the standard operations of algebra you learned in high-school math. Which of the following is a valid proof that $7 = 7$:

B

	I		II		III
1.	$7 = 7$	1.	$7 \neq 7$	1.	$0 = 0$
2.	$7 - 7 = 7 - 7$	2.	$7 - 7 \neq 7 - 7$	2.	$0 + 7 = 0 + 7$
3.	$0 = 0$ X	3.	$0 \neq 0$!FISHY	3.	$7 = 7$
$\rightarrow$	$7 = 7$	$\rightarrow$	$7 = 7$	$\rightarrow$	$7 = 7$

[A] I & II & III. [B] II & III. [C] I & II [D] I & III.

(g) Let $f(n) = \sum_{i=1}^{n} i$ and $g(n) = 2^{3 \log_2 n}$. What is the big-Oh relationship between f and g?

B

[A] $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

[B] $f(n) = O(g(n))$ and $g(n) \neq O(f(n))$.

[C] $f(n) \neq O(g(n))$ and $g(n) = O(f(n))$.

[D] $f(n) \neq O(g(n))$ and $g(n) \neq O(f(n))$.

$f(n) = \frac{1}{2} n(n+1)$

$g(n) = n^3$

$f \in o(g)$

$g \notin o(f)$

(h) You independently generate the ten bits of a binary sequence $b_1 b_2 \cdots b_{10}$ with $\mathbb{P}[b_i = 0] = \frac{1}{2}$. Compute the probability that the sequence is sorted from low to high. For example 0000111111 is sorted.

B

[A] $\frac{10}{1024}$

[B] $\frac{11}{1024}$

[C] $\frac{20}{1024}$

[D] $\frac{12}{1024}$

11 sequences sorted from high to low:

0...0 0...01 0...011 . . . . . 111...1

prob $= \frac{11}{2^{10}} = \frac{11}{1024}$

(i) x_1, x_2, x_3 are non-negative integers. Compute the number of different solutions to $x_1 + x_2 + x_3 = 100$. (For example two different solutions are $1 + 2 + 97 = 100$ and $97 + 1 + 2 = 100$.)

B

[A] 10302

[B] 5151

[C] 4949

[D] 5050

3 bins

$\binom{102}{2} = \frac{102 \cdot 101}{2} = 51 \cdot 101 = 51 + 5100 = 5151$

(j) For the automaton on the right, which input string is accepted? (Strings are processed from left to right.)

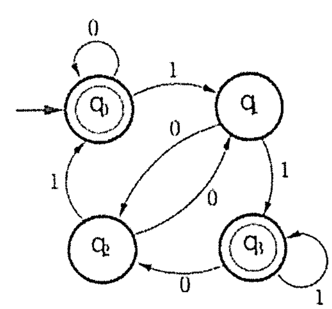
B

[A] 010101

[B] 0101011

[C] 01010110

[D] 010101100



2 Proofs

1. Prove that for all integers $n \geq 1$: $\quad n2^n \leq 3^n$

Proof By induction

Base Case $\quad P(1): \underbrace{1 \cdot 2^1}_{2} \leq 3^1$, ≤ 3 ✓ $\quad P(2): 2 \cdot 2^2 = 8 \leq 9 = 3^2$ ✓

P(1) and P(2) are true.

induction $\quad$ Assume $P(n): n2^n \leq 3^n$

Prove $P(n+1): (n+1)2^{n+1} \leq 3^{n+1}$

We know $n2^n \leq 3^n$

$$\therefore \quad (n+1)2^{n+1} = 2 \cdot n2^n + 2^{n+1}$$

$$\leq 2 \cdot 3^n + 2^{n+1} \quad \text{(IH)}$$

$$= 3^{n+1}\underbrace{\left(\frac{2}{3} + \left(\frac{2}{3}\right)^{n+1}\right)}_{\leq 1 \text{ for } n \geq 2}$$

$$\leq 3^{n+1} \quad \text{(for } n \geq 2)$$

when $n \geq 2$

$$\frac{2}{3} + \left(\frac{2}{3}\right)^{n+1} \leq \frac{2}{3} + \left(\frac{2}{3}\right)^3 = \frac{26}{27} \leq 1$$

$\therefore \quad P(n) \to P(n+1)$ for $n \geq 2$

$\boxed{P(1)}$ $\quad \boxed{P(2)} \to P(3) \to \cdots$

proved $\quad$ proved

By induction P(1) and P(n) for $n \geq 2$ are proved ∎

2. Prove that *every* odd natural number is the difference of two square numbers.

Proof using a general argument.

Let $n = 2k+1$ be a general odd number where $k \in \mathbb{Z}$

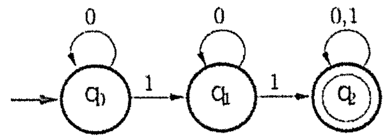
~~Consider the case as is positive~~

$$(k+1)^2 - k^2 = \cancel{k^2} + 2k + 1 - \cancel{k^2} = n$$

$\therefore \quad n = (k+1)^2 - k^2$ is a difference of squares.

3 Finite Automaton with a Random Input String

The automaton to the right processes a random binary string $b_1 b_2 \ldots b_n$ of length n generated as follows: you independently generate each bit b_i with $\mathbb{P}[b_i = 1] = p$ and $\mathbb{P}[b_i = 0] = 1 - p$. Show that the probability that the string is accepted is



$$\mathbb{P}[\text{ random input string is accepted }] = 1 - (1-p)^n - np(1-p)^{n-1}.$$

[Hints: (i) Figure out a simple property of a string for it to be accepted. (ii) Binomial distribution.]

A string is accepted if it has at least 2 1's.

~~Pf~~

$P[0 \text{ one's}] = (1-p)^n$

$P[1 \text{ one}] = np(1-p)^{n-1}$

$P[2 \text{ or more ones}] = 1 - P[0 \text{ ones}] - P[1 \text{ one}]$

$= 1 - (1-p)^n - n\,p(1-p)^{n-1}$

4 Probability and Expectation

(a) You independently roll 3 fair dice D_1, D_2, D_3 and let $S = D_1 + D_2 + D_3$ be the sum. Compute:

(i) $\mathbb{P}[S = 8]$

Condition on D_1

$D_1 =$ 1	$D_1 + D_2 = 7$	6		$\frac{1}{6}\cdot\left(\frac{6}{36}\right)$
2	$D_1 + D_2 = 6$	5		$\frac{1}{6}\left(\frac{5}{36}\right)$
3	$D_1 + D_2 = 5$	4		⋮
4	$D_1 + D_2 = 4$	3		⋮
5	$D_1 + D_2 = 3$	2		
6.	$D_1 + D_2 = 2$	1	possibility	$\frac{1}{6}\left(\frac{1}{36}\right)$

$P[S=8] = \frac{1}{6}\left[\frac{1+2+\cdots 6}{36}\right] = \frac{1}{\cancel{6}}\cdot\frac{\cancel{6}\cdot 7}{2\cdot 36} = \frac{7}{72}$

$\boxed{\frac{7}{72}}$

(ii) $\mathbb{P}[S = 8 \mid D_1 = 1]$

Given $D_1 = 1$, $D_2 + D_3 = 7$ has 6 possibilities

$\therefore P[S=8 \mid D_1 = 1] = \frac{6}{36} = \frac{1}{6}$.

OR $P[S=8 \mid D_1=1] = \frac{P[S=8 \wedge D_1 = 1]}{P[D_1 = 1]} = \frac{\cancel{\frac{1}{6}}\cdot\frac{6}{36}}{\cancel{\frac{1}{6}}}$

$= \frac{1}{6}$

$\boxed{\frac{1}{6}}$.

(iii) Compute the expectation and variance of S.

$E[D_1+D_2+D_3] = E[D_1] + E[D_2] + E[D_3] = 3\cdot E[D_1] = 3\cdot\frac{(1+2+\cdots+6)}{6} = \frac{3\cdot\cancel{6}\cdot 7}{2\cdot\cancel{6}} = \boxed{\frac{21}{2}}$

$Var(D_1+D_2+D_3) = Var(D_1) + Var(D_2) + Var(D_3) = 3\cdot Var(D_1)$

$E[D_1^2] = \frac{1}{6}[1^2+2^2+\cdots 6^2] = \frac{1}{6}[1+4+9+16+25+36] = \frac{1}{6}\cdot[10+20+61] = \frac{91}{6}$

$Var(D_1) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{364-294}{24} = \frac{70}{24} = \frac{35}{12}$.

$\therefore\ Var(S) = 3\cdot\frac{35}{12} = \boxed{\frac{35}{4}}$

(b) You toss a fair coin independently until you get two heads *in a row*. Let X be the number of tosses. Compute $\mathbb{E}[X]$ using the law of total expectation:

(i) Consider the 3 cases T, HT, HH for how the tosses may start and show that

$$\mathbb{E}[X] = \tfrac{1}{2}(1 + \mathbb{E}[X]) + \tfrac{1}{4}(2 + \mathbb{E}[X]) + \tfrac{1}{2}.$$

$E[X] = E[X \mid T]P[T] + E[X \mid HT]P[HT] + E[X \mid HH].P[HH]$.

$E[X \mid T] \Rightarrow$ restart $= 1 + E[X]$. $\quad P[T] = 1/2$

$E[X \mid HT] \to$ restart with 2 $= 2 + E[X]$. $\quad P[HT] = \frac{1}{4}$

$E[X \mid HH] = 2$. $\quad P[HH] = \frac{1}{4}$

$= \frac{1}{2}(1+E[X]) + \frac{1}{4}(2+E[X]) + \frac{1}{4}\cdot 2$.

$= \frac{1}{2}(1+E[X]) + \frac{1}{4}(2+E[X]) + \frac{1}{2}$

(ii) Use (i) to show that $\mathbb{E}[X] = 6$.

Solve for $E[X]$: $E[X] = \frac{1}{2} + \frac{1}{2}E[X] + \frac{1}{2} + \frac{1}{4}E[X] + \frac{1}{2} = \frac{3}{2} + \frac{3}{4}E[X]$

$\to \frac{1}{4}E[X] = \frac{3}{2} \to$ $\boxed{E[X] = 6.}$

5 Context Free Grammars

This problem is about the language $\mathcal{L}$ generated by the CFG:

$$\begin{aligned} S &\to 1T \mid 0T \\ T &\to 1T1 \mid 0T0 \mid \epsilon \end{aligned}$$

(a) Is the string 1010010 in $\mathcal{L}$? If yes then give a derivation or parse tree; if no then explain why.

Yes:

$S \longrightarrow 1T \longrightarrow 1\,0T0 \longrightarrow 1\,01T1\,0 \longrightarrow 1\,010T010 \longrightarrow 1010010.$

(b) Prove that the length of every string in $\mathcal{L}$ is odd.

Every string in $\mathcal{L}$ starts with the derivation $S \to 1T$ or $S \to 0T$

$\therefore$ The strings derived are 1 longer than the strings in T.

It suffices to prove that the strings in T are even.

Proof By induction on derivation length. [Strong Induction].

Base $T \to \varepsilon$ (length 1 derivation) ✓

length 0.

Induction: Assume all derivations of length up n give even length.

Consider derivation of length $n+1$

The derivation starts $T \to 0T0$ or $T \to 1T1$

$T \xrightarrow{*} x$, $T \xrightarrow{*} y$

the derivations $T \xrightarrow{*} x$ and $T \xrightarrow{*} y$ are length n derivations

$\therefore$ by the induction hypothesis $|x| =$ even $|y| =$ even.

$\therefore |0x0| = 2 + |x| =$ even or $|1y1| = 2 + |y| =$ even.

In either case the length is even.

By induction every derivation in T gives even length.

$\therefore$ Every derivation in S gives ~~odd~~ length.

6 Turing Machine

(a) What is the difference between a Turing-recognizable language and a Turing-decidable language?

Turing recognizable: TM halts on YES-set with accept
TM halts or loops forever on NO-set

Turing decidable: TM halts on YES-set with accept
TM halts on No-set with reject.

(b) Consider the arithmetic task of squaring, which corresponds to the language $\mathcal{L} = \{0^n \# 0^{n^2} | n \geq 1\}$.

(i) Circle the simplest model of computing that you think solves the problem $\mathcal{L}$:

Finite Automaton Context Free Grammar Turing Machine

(ii) Give your machine from (i) that solves $\mathcal{L}$ (for a TM, a high level description will do).

① Check input format is $0^* \# 0^*$.

② Mark a ~~each~~ zero left of # with x.
for each zero left of #, mark it with ✓ and mark a zero right of #
When all zeros left of # have ✓, unmark all ✓ from the left zeros.

③ Repeat as long as there are left zeros.

④ If run out of right zeros, reject
if run out of left zeros, check no unmarked right zeros.
if so accept
else reject.