# FINAL: 180 Minutes

Last Name: Solutions

First Name: ____________

RIN: ____________

Section: ____________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.

**NO COLLABORATION or electronic devices. Any violations result in an F.**

**NO questions** allowed during the test. Interpret and do the best you can.

**GOOD LUCK!**

| 1 | 2 | 3 | 4 | 5 | Total |
|---|---|---|---|---|---|
| | | | | | |
| 200 | 40 | 40 | 40 | 40 | 350 |

(10 bonus points)

## 1 Circle at most one answer per question. 10 points for each correct answer.

(1) The negation of "Every student is a friend of some other student" is

- [A] Some student has a friend who is a student.
- [B] Some student is a friend of all students.
- [C] Some student is not a friend of some other student.
- [D] Some student is not a friend of all other students.
- [E] Some student has no friends.

D

$\forall s : [\exists x : F(s,x)]$

$\sim \forall s : [\exists x : F(s,x)]$

$\equiv \exists s : \sim \exists x : F(s,x)$

$\equiv \exists s : \forall x : \sim F(s,x).$

(2) Estimate $2^1 \times 2^2 \times 2^3 \times \cdots \times 2^{20} = \prod_{i=1}^{20} 2^i$.

- [A] $1.65 \times 10^{61}$
- [B] $1.65 \times 10^{63}$
- [C] $1.65 \times 10^{65}$
- [D] $1.65 \times 10^{67}$
- [E] $1.65 \times 10^{69}$

B

$2^{1+2+\cdots+20} = 2^{20\cdot 21/2} = 2^{210} \approx (2^{20})^{10.5}$

$2^{20} \approx 10^6$ $\quad \frac{210}{20} = 10.5$ $\quad \approx (10^6)^{10.5}$

$10.5 \times 6 = 63$ $\quad \approx 10^{63}.$

(3) What is the *most accurate* order relation between $2^n$ and $e^n$?

- [A] $2^n \in o(e^n)$.
- [B] $2^n \in O(e^n)$.
- [C] $2^n \in \Theta(e^n)$.
- [D] $2^n \in \Omega(e^n)$.
- [E] $2^n \in \omega(e^n)$.

A

$2^n \in o(e^n)$. because $\frac{2^n}{e^n} = \left(\frac{2}{e}\right)^n \to 0.$

(4) $f(n)$ satisfies the recurrence $f(0) = 1$; $f(n) = nf(n-1)$. Which order relationship describes $f$.

- [A] $f \in \Theta(2^n)$.
- [B] $f \in O(2^n)$.
- [C] $f \in o(2^n)$.
- [D] $f \in \Theta(n^n)$.
- [E] $f \in o(n^n)$.

E

$f(n) = n!$ $\quad n! \in o(n^n)$.

because $\frac{n!}{n^n} \leq \frac{n^{n-1}}{n^n} = \frac{1}{n} \to 0.$

(5) What is the greatest common divisor of 756 and 840?

[A] 12.

[B] 28.

[C] 63.

D. [D] 84.

[E] 189.

$\gcd(756, 840) = \gcd(84, 756)$
$= \gcd(0, 84)$
$= 84.$

← Euclid's Algorithm.

$84 \overline{)756}$ 009 rem 0

(6) What is the minimum number of colors needed to color the graph on the right?

[A] 2.

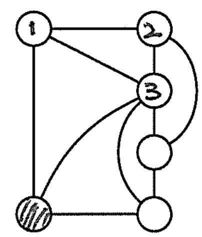
[B] 3.

C [C] 4.

[D] 5.

[E] 6.

start with the clique which needs 1, 2, 3. This forces the shaded node to be 4.




(7) On the right is the $4 \times 12$ grid graph. What is the average degree of a node?

[A] 3.

[B] $3\frac{1}{4}$.

C [C] $3\frac{1}{3}$.

[D] $3\frac{1}{2}$.

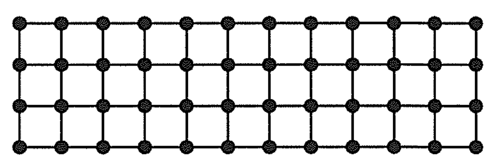
[E] $3\frac{2}{3}$.

Total edges $= 3 \times 12 + 11 \times 4$
$= 36 + 44 = 80$

Total nodes $= 4 \times 12 = 48$

$\therefore$ Sum of degrees $= 2 \times |E|$
$= 160$

$\therefore$ Average degree $= \frac{160}{48} = \frac{40}{12} = 3\frac{4}{12} = 3\frac{1}{3}$



(8) Shirts come in 6 colors. 4 students are in a row. You must assign shirts to the students, and two students standing next to each other cannot get the same color shirt. In how many ways can you do this?

[A] $\binom{9}{3}$.

[B] $6 \times 5 \times 4 \times 3$.

[C] $\binom{6}{4}$.

D [D] $6 \times 5^3$.

[E] $6^4$.

$6 \times 5 \times 5 \times 5$
$= 6 \cdot 5^3$

first student 6 possibilities
each next student 5 possibilities.

(9) Pokemon characters have a 4 digit serial number, e.g. 0255. A pokemon is defective if any digit is repeated (e.g. 0255 is defective). *Approximately* what fraction of the possible serial numbers are defective?

[A] 0.

[B] 0.25.

[C] 0.5.

[D] 0.75.

[E] 1.

\# serial #'s = $10^4$

non-defective = $10 \times 9 \times 8 \times 7$

defective = $10^4 - 10 \times 9 \times 8 \times 7$

fraction defective = $1 - \frac{\cancel{10} \times 9 \times 8 \times 7}{\cancel{10^4}\, 10^3} = 1 - \frac{9 \times 8 \times 7}{10^3} = 1 - \frac{504}{1000}$

$\approx 0.5$

C

(10) A senate committee of 10 senators must pick a president. 3 candidates will be proposed from the 10 senators, and everyone votes. In how many ways can the 3 candidates be chosen.

[A] 1000.

[B] 720.

[C] 120.

[D] 10!

[E] $\frac{10!}{3!}$.

$\binom{10}{3} = \frac{10!}{7!\,3!} = \frac{\cancel{10}^{5} \times \cancel{9}^{3} \times 8}{\cancel{3} \times \cancel{2} \times 1} = 120.$

C.

(11) Three integers $z_1, z_2, z_3$ satisfy $0 \leq z_1 \leq z_2 \leq z_3 \leq 6$ (the sequence is non-decreasing and bounded between 0 and 6). How many such sequences are there?

[A] 28.

[B] 42.

[C] 84.

[D] 165.

[E] 168.

Brute force: $\sum_{i=0}^{6} \sum_{j=i}^{6} \sum_{k=j}^{6} 1$

Quicker: $x_1 = z_1 \quad x_2 = z_2 - z_1 \quad x_3 = z_3 - z_2. \qquad x_i \geq 0.$

$x_1 + x_2 + x_3 = \cancel{z_1} + \cancel{z_2} - \cancel{z_1} + z_3 - \cancel{z_2} = z_3 \leq 6.$

$\therefore x_1 + x_2 + x_3 \leq 6$

$\therefore x_1 + x_2 + x_3 + x_4 = 6 \qquad x_i \geq 0$

$\rightarrow$ binary seq 6 zeros, 3 1's $\rightarrow \binom{9}{3} = \frac{\cancel{9}^{3} \times \cancel{8}^{4} \times 7}{\cancel{3} \times \cancel{2}} = 12 \times 7 = \underline{84}.$

C.

(12) You are thinking of a graph with 4 nodes Ⓐ Ⓑ Ⓒ Ⓓ. How many such graphs are there?

[A] 24.

[B] 64.

[C] 81.

[D] 256.

[E] 4096.

\# edges = $\binom{4}{2} = 6$

Each edge has two choices: there or not $\therefore 2^6$ graphs

$2^6 = 64$

B

(13) $\mathbf{X}, \mathbf{Y}$ are random variables (not necessarily independent) and $\mathbf{Z} = a\mathbf{X} + b\mathbf{Y}$. What is $\mathbb{E}[\mathbf{Z}]$?

A $a\,\mathbb{E}[\mathbf{X}] + b\,\mathbb{E}[\mathbf{Y}]$

B $a^2\,\mathbb{E}[\mathbf{X}] + b^2\,\mathbb{E}[\mathbf{Y}]$

C $(a+b)(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}])$

D $a(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}]) + b(\mathbb{E}[\mathbf{X}] + \mathbb{E}[\mathbf{Y}])$

E None of the above are true in general.

Linearity of expectation.

A

(14) This test has 20 multiple choice questions, each with 5 possible choices. If you answer questions randomly, what is the expected number of questions you get correct?

A 3

B 4

C 5

D 6

E 10

$P[\text{correct}] = \frac{1}{5}$

$X = X_1 + \cdots X_{20}$

$E[X] = \underbrace{E[X_1]}_{\frac{1}{5}} + \cdots + \underbrace{E[X_{20}]}_{1/5} = \frac{20}{5} = 4.$

B

(15) About 1 in a 1000 people have Coeliac disease. The test for Coeliac randomly makes a mistake 5% of the time (95% accuracy). You tested positive. *Approximately* what are the chances you have Coeliac?

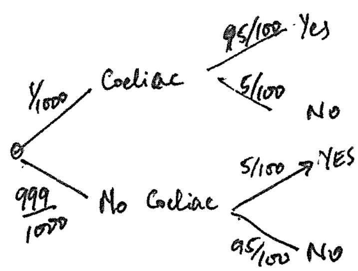
A 0.2%

B 2%

C 20%

D 50%

E 95%




want $P[\text{Coeliac} \mid \text{YES}] = \frac{P[\text{Coeliac} \wedge \text{YES}]}{P[\text{YES}]}$

$= \frac{\frac{1}{1000} \times \frac{95}{100}}{\frac{1}{1000} \times \frac{95}{100} + \frac{999}{1000} \times \frac{5}{100}} = \frac{95}{95 + 999 \times 5}$

$\approx \frac{100}{5000} = \frac{1}{50} = 2\%.$

B

(16) A random binary string $b_1 b_2 \ldots b_{10}$ of 10 bits is the input to the automaton. What is the probability that the string is accepted?

A $\frac{2}{1024}$

B $\frac{45}{1024}$

C $\frac{56}{1024}$

D $\frac{90}{1024}$

E $\frac{512}{1024}$

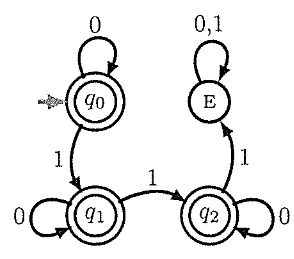
Accept if 0, 1 or 2 1's; Reject otherwise

# sequences is $\binom{10}{0} + \binom{10}{1} + \binom{10}{2}$

$1 + 10 + \frac{10 \times 9}{2} = 56.$

$\therefore P[\text{accept}] = \frac{56}{2^{10}} = \frac{56}{1024}.$

C



(17) What is a computing problem?

E

[A] A Person.

[B] An automaton (machine which transitions between states as it reads the input).

[C] An automaton with stack memory.

[D] An automaton with random access memory.

[E] A set containing finite binary strings.

(18) The computing problem $\mathcal{L}$ = {strings with an even number of 1s} can be solved by:

(I) DFA. (II) CFG. (III) Turing Machine.

DFA solves → CFG and TM do too.

A

[A] I,II,III

[B] I,III

[C] II,III

[D] III only

[E] None of these models of computing

(19) The computing problem $\mathcal{L}$ = {strings corresponding to programs which HALT} can be solved by:

(I) DFA. (II) CFG. (III) Turing Machine.

ultimate-debugger does not exist ∴ undecidable.

E

[A] I,II,III

[B] I,III

[C] II,III

[D] III only

[E] None of these models of computing

(20) A DFA has *two* states a start state $q_0$ and a second state $q_1$. The DFA is described by a list of its accept states and a list of its transition instructions. The order in which you list the accept states and the transition instructions does not matter. We draw a DFA as a graph with nodes $q_0, q_1$ and add a directed arrow for each transition instruction (the accepting states have double circles).

How many different DFA's are there with two states? (*Different* DFA's *can* have the same (YES)-set)

E

[A] 4.

[B] 8.

[C] 16.

[D] 32.

[E] 64.

Accept states: subset of 2 states → $2^2 = 4$ possible subsets.

Each state $\underbrace{\text{has an arrow for 0,1}}_{\text{4 arrows}}$ and each arrow can $\underbrace{\text{go to either state}}_{\text{2 options}}$

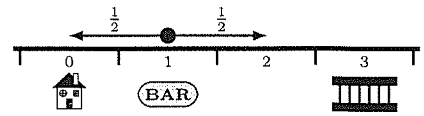
$\Rightarrow 2^4$ choices for arrow destinations

$\therefore$ # DFA's $= 4 \times 2^4 = 64$

accept states transitions

## 2 Random Walk

A drunk leaves the bar (at position 1), and takes independent steps: left (L) with probability $\frac{1}{2}$ or right (R) with probability $\frac{1}{2}$. The drunk stops when he reaches home or the bar. Compute the *expected* number of steps the drunk makes.




Let $X$ = # of steps

Total Expectation:

$$E[X] = \underbrace{E[X|L]}_{1}\underbrace{P[L]}_{\frac{1}{2}} + \underbrace{E[X|RR]}_{2}\underbrace{P[RR]}_{\frac{1}{4}} + \underbrace{E[X|RL]}_{2+E[X]}\underbrace{P[RL]}_{\frac{1}{4}}$$

$$= \frac{1}{2} + \frac{2}{4} + (2+E[X])\cdot\frac{1}{4}$$

$$= \frac{3}{2} + \frac{E[X]}{4}$$

$$\longrightarrow \quad \frac{\cancel{3}E[X]}{4} = \frac{\cancel{3}}{2} \longrightarrow \boxed{E[X] = 2}$$

---

Brute Force Method: Possible Outcomes.

$$(RL)^i L \quad \text{and} \quad (RL)^i RR$$

$$P = \left(\tfrac{1}{2}\right)^{2i+1} \qquad P = \left(\tfrac{1}{2}\right)^{2i+2}.$$

$$E[X] = \sum_{i=0}^{\infty}(2i+1)\left(\frac{1}{2}\right)^{2i+1} + \sum_{i=0}^{\infty}(2i+2)\left(\frac{1}{2}\right)^{2i+2}$$

$$= \sum_{i=\cancel{0}}^{\infty} i\left(\frac{1}{2}\right)^{i}$$

$$= \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$

$$\longrightarrow \boxed{E[X] = 2}$$

$$S = \sum_{i=1}^{\infty} i a^i = a + 2a^2 + 3a^3 + \cdots$$

$$aS = a^2 + 2a^3 + \cdots$$

$$S - aS = a + a^2 + a^3 + \cdots = \frac{a}{1-a}$$

$$S(1-a) = \frac{a}{1-a} \longrightarrow S = \frac{a}{(1-a)^2}.$$

## 3 Induction

(a) $G(n) = 1;$ $\quad$ Prove that $\underbrace{G(n) = \frac{1}{n}}_{P(n)}$ for integer $n \geq 1$.

$G(n) = G(n-1)\left(1 - \frac{1}{n}\right)$ for $n > 1$;

Base Case $P(1)$ $\cancel{\triangleq}$ $G(1) = 1 = \frac{1}{n}$.

Induction Assume $P(n) = \frac{1}{n}$

$$G(n+1) = G(n)\left(1 - \frac{1}{n+1}\right) = \frac{1}{n} \cdot \left(1 - \frac{1}{n+1}\right)$$

$$= \frac{1}{\cancel{n}} \frac{\cancel{n+1-1}}{n+1}$$

$$= \frac{1}{n+1}$$

$\rightarrow$ $P(n+1)$ is true $\quad [P(n) \rightarrow P(n+1)]$.

$\therefore$ $P(n)$ is true for $n \geq 1$.

(b) The $n$th Harmonic number is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that $\underbrace{H_1 + H_2 + \ldots + H_n = (n+1)H_n - n}_{P(n)}$.

Base Case. $P(1)$: $H_1 = 1 = (1+1) \cdot 1 - 1$ ✓.

Induction Assume $P(n)$: $H_1 + H_2 + \cdots H_n = (n+1)H_n - n$

Prove $P(n+1)$: $H_1 + \ldots H_n + H_{n+1} = (n+1)H_n - n + H_{n+1}$ [Induction Hypothesis].

$$H_n = H_{n+1} - \frac{1}{n+1} \rightarrow \quad = (n+1)\left[H_{n+1} - \frac{1}{n+1}\right] - n + H_{n+1}$$

$$= (n+1)H_{n+1} - 1 - n + H_{n+1}$$

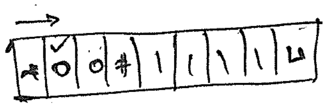
$$= (n+2)H_{n+1} - (n+1)$$

$\longrightarrow$ $P(n+1)$ is true $\therefore$ $P(n) \rightarrow P(n+1)$

and $P(n)$ is true by induction for all $n \geq 1$.

## 4 Turing Machine

Give a high-level description of a Turing Machine that solves the problem $\mathcal{L} = \{0^n\#1^{n^2} \mid n \geq 0\}$ (squaring). (You may find it useful to illustrate how your TM works on 00#1111.)



① Check the string format.

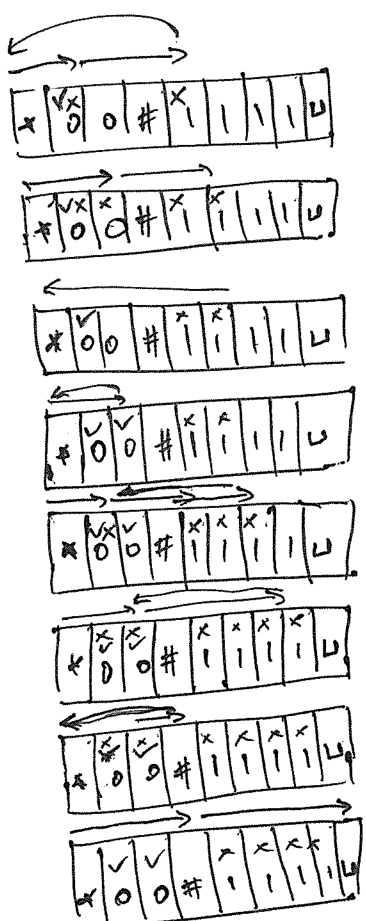
② For each zero (mark with a ✓)

Repeat

from * Move right and mark (x) first zero
move right and mark (x) 1
(if no 1 reject.
If no zero to mark x, move L removing x.
return to *

marks n 1's

③ If no zero to mark with ✓
move right to check no unmarked 1.



ACCEPT .

## 5 [Hard] Unsolvable Problems

*Prove*: There is an undecidable computing problem which is a subset of $\{1\}^*$. *[Hint: Infinite binary strings.]*

$\boxed{\text{Method 1}}$ Subsets of $\{1\}^*$ are uncountable; TM's countable $\therefore$ there must be some subset not solved by a TM; this subset is undecidable.

Counting Argument

$\{1\}^* = \{1^0, 1^1, 1^2, 1^3, \ldots\}$

List the TM's for which $\mathcal{L}(M) \subseteq \{1\}^*$

| | $1^0$ | $1^1$ | $1^2$ | $1^3$ | $1^4$ ... |
|---|---|---|---|---|---|
| $\mathcal{L}(M_1)$ | 1 | 0 | 0 | 1 | ... |
| $\mathcal{L}(M_2)$ | 0 | 1 | 1 | 0 | ... |
| $\mathcal{L}(M_3)$ | 1 | 0 | 0 | 1 | ... |
| ⋮ | | | | | |

← 1 indicates which string is in $\mathcal{L}(M_i)$

$\mathcal{L}(M_i) \longrightarrow$ infinite binary string

We have a list of infinite binary strings.
By Cantor diagonalization, some infinite string is not in this list.

That is: The language corresponding to this string is not decided by a TM.

$\boxed{\text{Method 2}}$ Reduction

List TM's $M_1\ M_2\ M_3\ M_4 \ldots$

Infinite Binary Sequence $1\ \ 0\ \ 1\ \ 0$

$M_i$ Halts

$M_i$ does not halt.

So there is some function which given $M$ computes $i$, the position in the list.

This "halting" binary sequence defines a subset of $\{1\}^*$. Call it $\mathcal{L}$.

$$1^i \in \mathcal{L} \longleftrightarrow M_i \text{ halts.}$$

Let $A$ be a decider for $\mathcal{L}$.

$$\therefore\ A(1^i) = \begin{cases} \text{accept} \longleftrightarrow 1^i \in \mathcal{L} \longleftrightarrow M_i \text{ halts} \\ \text{reject} \longleftrightarrow 1^i \notin \mathcal{L} \longleftrightarrow M_i \text{ infinite loops.} \end{cases}$$

Given $M \xrightarrow{\text{compute}} i \xrightarrow{\text{run}} A$ on $1^i$ ← this is a decider for

$\mathcal{L}_{HALT} = \{ \langle M \rangle \mid M \text{ HALTS} \}$

But $\mathcal{L}_{HALT}$ is undecidable $\therefore$ ~~$\mathcal{L}_{HALT}$~~ $A$ does not exist $\therefore \mathcal{L}$ is undecidable.